Thomas Aquinas' Mathematical Realism

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To Maria
"Da mihi basia mille, deinde centum."

Preface

Anyone who studies pre-Enlightenment philosophy will be struck by an obvious difference—it begins with knowing, whereas doubt seems to have *characterized* efforts at philosophizing from Descartes to our day. That this was not—at some point—*universally* so was due in large part to mathematicians, or it at least *seems* that mathematics was the last area to be challenged. But challenged it was—and philosophers of mathematics are still reeling from it. We are very much in agreement with the spirit of mathematical realism, which insists that there still are necessary truths to be found in the mathematical realm—which is also the realm of the real. It is one thing to *prescind* from the question whether there is a real foundation for mathematical truths, and quite another to deny that basis altogether.

This book is a contribution to the effort of restoring confidence in the mind's capacity to know. The sense that $2 + 2 = 4$ is an eternal verity is still *out there*—which is to say that the tendency to anti-realism has not yet become a pervasive frame-of-mind. It is our contention that some of the difficulties inclining contemporary philosophers of mathematics to anti-realism can be illuminated, and possibly avoided, by a careful and judicious return to earlier views and methods. We see in the philosophies of Aristotle and Thomas Aquinas much of value—both in itself and with a view to clearing up some of the confusion surrounding mathematical foundations. We hope you will find it worth your efforts.

I most gratefully acknowledge the help I have received in writing this book—it is the culmination of a lifetime of wondering. To begin at the beginning, many thanks to my tutors at Thomas Aquinas College, who

instilled in me a love for the great works and the truths which can be found within, especially my mentor and friend, Thomas Aquinas McGovern, S.J., who showed me more clearly than anyone how brilliant his namesake—albeit a Dominican—could be, and Dr. Ronald Richard, who introduced me to the wonders of modern mathematics. I wish also to thank Dr. Patrick Lee, who directed me through the writing of my dissertation at the Center for Thomistic Studies—a man who well knew the enduring universality of Thomas Aquinas' thought. Among my colleagues in the philosophy department here at Benedictine College I would be remiss not to mention Dr. Donald Scholz, whose love for mathematical philosophy was and has been an inspiration, and Dr. James Madden, who persuaded me that a *lingua franca* does indeed exist between traditional Thomism and the analytic school. To my family and departmental colleagues generally I also wish to convey my gratitude. Though I would not go so far as to call it *distracted*, my work on this book over the years has meant some sacrifice on their part, from a father whose grapplings with a subtle problem or distinction did not wait, to late reports and recommendations from a department chair. Thank you all.

Atchison, KS Jean W. Rioux

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CHAPTER 1

Introduction

What Henry Veatch once observed of philosophy as a whole seems applicable to the philosophy of mathematics in our day—*thou art in a parlous state!*[1] While mathematical realism was—more or less—presumed by those either engaged in, or talking about, mathematics in centuries past, in modern times questions about why we are compelled to mathematical conclusions and upon what basis this occurs have given rise to a new, critical discipline within philosophy as a whole. This is not to say that there have not always been philosophers of mathematics of a sort—one would not expect philosophers with such widely ranging interests as Plato and Aristotle, on the one hand, and Sextus Empiricus, on the other, to have left such questions untouched, nor did they.[2] Nevertheless, it is not uncommon to hear of an ongoing crisis in mathematics today, something which earlier thinkers would have found puzzling. Why the difference?

The development of non-Euclidean geometries in the early nineteenth century raised fundamental questions for mathematicians and

[1] Henry Babcock Veatch, "Philosophy, thou art in a parlous state!" in *Thomistic Papers 1*, ed. Victor B. Brezik, C.S.B. (Houston: Center for Thomistic Studies, 1984), 9–44.

[2] Platonism is among the oldest and most hotly debated philosophies of mathematics.

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J. W. Rioux, *Thomas Aquinas' Mathematical Realism*,
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philosophers of the twentieth century and beyond.[3] While Euclid's *parallel postulate* had been a matter of debate for centuries, only through the work of Gauss, Bolyai, Lobachevski, and others did non-Euclidean geometries gain acceptance among mathematicians in general.[4] Importantly, mathematicians themselves seem to have gradually distinguished their own work from that of their more philosophical colleagues. Aristotle himself adverted to this difference, when he said that denying the actual infinite does not hamper mathematicians at all, as they do not need it.[5] One can find corresponding sentiments expressed by mathematicians and philosophers throughout the history of mathematics.[6] So, when I say that a crisis occurred, it is not a crisis necessarily recognized as such by mathematicians: most of the anguish surrounding the downfall of *the most certain of the sciences* seems to have been on the part of philosophers and not mathematicians, which latter went about their work as they have always done. Indeed, mathematicians began to enjoy a new sort of freedom in their investigations and in these unexpected avenues for exploration and application. However much the development of non-Euclidean geometries led to troubling metaphysical questions about the nature of the universe, mathematicians and scientists found in them opportunities for

[3] One source of this view is Morris Kline, for whom "no more cataclysmic event has ever taken place in the history of all thought." Morris Kline, *Mathematics in Western Culture* (London: George Allen and Unwin, 1956) 428. To be fair, others, for example, Hillary Putnam, are not so sure the crisis is real. Still, real or not, the *perception* of one occasioned a great deal of foundational work by mathematicians and philosophers alike and a splintering of that part of philosophy into a dozen or more schools of thought. See the excellent and varied selection of readings in modern mathematical philosophy compiled by Benacerraf and Putnam. Paul Benacerraf and Hillary Putnam, eds., *Philosophy of Mathematics: Selected Readings* (Cambridge: Cambridge University Press, 1983).

[4] Morris Kline observes that the work of the later non-Euclideans, Bolyai and Lobachevsky, did not begin to receive attention until posthumously published accounts defending non-Euclidean systems, from the hand of the better-known and already-respected Gauss, came to light. Kline, *Mathematics in Western Culture*, 441–515.

[5] "This argument [against an actual infinite] does not deprive mathematicians of their study … for, in fact, they do not need the [actual] infinite (since they do not use it)." Aristotle, *Physica (Oxford: Oxford University Press,* 1982), III 7207b27–30. οὐκ ἀφαιρεῖται δ' ὁ λόγος οὐδὲ τοὺς μαθηματικοὺς τὴν θεωρίαν … οὐδὲ γὰρ νῦν δέονται τοῦ ἀπείρου (οὐ γὰρ χρῶνται). [This translation and all subsequent translations of Greek passages from Aristotle and Latin passages from Thomas Aquinas are my own.]

[6] One clear example of this divide is found in Heyting's account of a discussion between a classical mathematician and an intuitionist regarding the twin primes problem. Arend Heyting, "Disputation," in Benacerraf and Putnam, *Philosophy of Mathematics*, 66–76.

solving real problems which the idea of such spaces made possible. In a similar vein, the development of transfinite numbers made it possible for mathematicians to imagine fruitful comparisons between and distinctions among infinite sets, despite the opposition thinkers like Georg Cantor met from neo-scholastic philosophers and even some of his own colleagues.

What we propose in this volume is a reconsideration of one mathematical philosophy which seems to have been marginalized or even severely misunderstood along the way: that of the medieval philosopher Thomas Aquinas. We think some of the confusion confronting modern discussions of the crisis in mathematical foundations can be averted by a careful review and application of Aquinas' account of human knowledge in general and of the sciences of mathematics in particular.[7] We plan to show that this account supplies possible avenues for decisive solutions to the difficulties which have dogged modern philosophies of mathematics just as they began to develop.

Aquinas' own mathematical education was apparently unremarkable.[8] It is his reflections upon the nature of mathematics itself which begin to distinguish him—reflections taking us directly back to the man he named *The Philosopher*. It behooves us, then, to begin our forward-looking review with a careful account of the key features of Aristotle's own philosophy of mathematics. This, in turn, requires that we begin with a brief account of Plato, whose own school was reputed to have displayed a Dante-esque warning to the uneducated: μηδείς ἀγεωμέτρητος εἰσίτω μου τὴν στέγην.[9]

The crisis in mathematical foundations has two facets: the nature of mathematical *reasoning* and the classification of mathematical *objects*.[10] We

[7] Of course, even his use of the word *sciences* will need to be qualified, given the view of science dominating the minds of mathematical and philosophical thinkers of the last two or so centuries.

[8] His biographers barely mention what mathematics the teenage Aquinas would have learned in the *studium generale* at Naples, except to say that he learned quickly and well. Weisheipl suggests that he would have read Boethius' *De Arithmetica*, Euclid's *Elements* I–VI, the *Almagest* of Ptolemy, and Boethius' *De Musica*. James Weisheipl, O.P., *Friar Thomas D'Aquino* (Garden City, New York: Doubleday, 1974) 13–7.

[9] "Let no one ignorant of geometry enter within."

[10] Analytical vs. synthetic would be a way to divide the first, for example, while the various realisms and anti-realisms is a possible division of the second. Of note, in his articles "What Numbers Could Not Be" and "Mathematical Truth," Paul Benacerraf presents realist mathematicians with a dilemma patterned after this same distinction, a dilemma which became the focal point for a further splintering of views in the last twenty-five years or so. In Benacerraf and Putnam, *Philosophy of Mathematics*, 272–94 and 403–20, respectively.

will begin our review of Aristotle with his account of the speculative and middle sciences (mathematics occupies a place within each grouping), followed by his account of mathematical objects. As these two considerations form the stable points of an organic account, we will also investigate Aristotle's understanding of how one arrives from object to science, here as elsewhere, which consideration brings us foursquare into questions of mathematical abstraction.

While Aquinas' own mathematical account is fundamentally Aristotelian, there are developments and qualifications which he himself brings to the question. One of these is his insistence upon mathematics as both science (one of three) and an art. Furthermore, Aquinas sees specifically mathematical abstraction as bearing upon what the human imagination brings to the picture. Finally, while Aristotle finds in his distinction between potency and act a solution to the *precision problem* of his day and ours, Aquinas gives us both the language and distinctions necessary to develop that into a full-blown solution. The key is his notion of *virtual* existence.

While sufficient for a historical account of mathematics in the peripatetic tradition, we bring a reconsideration of Aristotle and Aquinas to the modern mathematical view in order to fruitfully address the *parlous state* we mentioned earlier. If the confusion inherent in the differing and diverging accounts we face is to be at least diminished, perhaps it would be helpful to see what these men would have said about such difficulties as we face today. No doubt, some will think the introduction of such an account as this might only worsen the situation. We disagree and offer in its place a Thomistically based solution to problems arising in number theory, Kline's revolution in geometry, the difficulties inherent in a mathematics of the infinite, and the more recent controversies between realists and anti-realists. We think there is something to be gained in rejecting the type of *all or nothing* reasoning we see prevailing in the mathematical thought of our day. On the other hand, philosophy of mathematics is not the same as mathematics (both Aristotle and Aquinas would have included it under metaphysics) and we know that the present volume will not end the debate. Still, we think it is an important step in bringing the philosophy of mathematics back to a reasonable state of health, and if it achieves some part of that outcome, it will have been worth the writing of it.

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PART I

Mathematical Realism in Plato and Aristotle

CHAPTER 2

Plato on Mathematics and the Mathematicals

Platonism as Mathematical Realism

We cannot settle a controversy ranging from Plato the realist to Plato the idealist in this brief chapter on his mathematical views. Better to focus upon Plato as he has come to be understood by many mathematicians—there is some force in speaking of Plato as a realist among people for whom *mathematical realism* is nearly synonymous with *platonism*.[1]

Two *loci classici* for Plato on mathematics are the *Timaeus* and the *Republic*.[2]

In the *Timaeus*, Plato supplies us with an account of the mathematical structure found in natural things. Indeed, as Timaeus relates, though they are not originally natural, nature is *shot through* with mathematicals. Literally following Socrates' account of the ideal *polis* from (presumably) the *Republic*, and with a view to extending the political account given there, Critias relates the history of the fabled state of Atlantis and its ancient commerce with Athens. At one point, however, Critias passes the

[1] Jonathan Lear makes much of a helpful distinction between *platonism*, positing the reality of abstract mathematical objects independently of our knowing them, and *Platonism*, Plato's historical account of mathematics. Jonathan Lear, "Aristotle's Philosophy of Mathematics," *The Philosophical Review* 90, no. 2: 161–92.

[2] While the *Meno* also explicitly includes a discussion of mathematics, it has more to do with general epistemology and not mathematics proper. We will be looking more closely at passages from this dialog in a later chapter.

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J. W. Rioux, *Thomas Aquinas' Mathematical Realism*,
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role of narrator on to Timaeus, who reaches back even further, to the origins of the natural world and humankind themselves. As he says:

> The city and citizens, which you yesterday described to us in fiction, we will now transfer to the world of reality. It shall be the ancient city of Athens, and we will suppose that the citizens whom you imagined, were our veritable ancestors, of whom the priest spoke; they will perfectly harmonise, and there will be no inconsistency in saying that the citizens of your republic are these ancient Athenians. Let us divide the subject among us, and all endeavour according to our ability gracefully to execute the task which you have imposed upon us.[3]

The task, we learn earlier, is to *bring to life* the abstract republic of the previous day's discourse—to "transfer [it] to the world of reality."[4] Arguably, Socrates is asking for, and Critias and Timaeus are supplying, the very thing Timaeus eventually credits to God Himself: transferring the eternal and unchangeable world of forms to the imperfect and changeable world of nature. Beginning with the distinction between these two worlds—and in parallel with the divided line from the *Republic*—Timaeus supplies an account of God's introduction of order and beauty to the disordered and changeable world of earth, air, fire, and water. Importantly, for our purposes, this order includes mathematical proportions and geometrical shapes—again, the realm of the necessary and unchangeable is entering the changeable, material realm.

If we take this account to represent Plato's view, the mathematicals are immaterial realities, eternal and unchangeable, and serve as patterns whereby God has infused order within the material universe. As to mathematics, as Timaeus famously says, "As becoming is to being, so is belief to truth."[5] The opinions we have of the natural universe (again, "the likely story"[6] we have concocted—we, including Timaeus and his own account

[3] Plato, "Timaeus", trans. Benjamin Jowett in *The Collected Dialogues of Plato*, eds. Edith Hamilton and Huntington Cairns (New Jersey: Princeton University Press, 1978), 26d, 1160. τοὺς δὲ πολίτας καὶ τὴν πόλιν ἣν χθὲς ἡμῖν ὡς ἐν μύθῳ διήεισθα σύ, νῦν μετενεγκόντες ἐπὶ τἀληθὲς δεῦρο θήσομεν ὡς ἐκείνην τήνδε οὖσαν, καὶ τοὺς πολίτας οὓς διενοοῦ φήσομεν ἐκείνους τοὺς ἀληθινοὺς εἶναι προγόνους ἡμῶν, οὓς ἔλεγεν ὁ ἱερεύς. πάντως ἁρμόσουσι καὶ οὐκ ἀπᾳσόμεθα λέγοντες αὐτοὺς εἶναι τοὺς ἐν τῷ τότε ὄντας χρόνῳ. κοινῇ δὲ διαλαμβάνοντες ἅπαντες πειρασόμεθα τὸ πρέπον εἰς δύναμιν οἷς ἐπέταξας ἀποδοῦναι.

[4] Ibid.

[5] Ibid., 29c, 1162. ὅτιπερ πρὸς γένεσιν οὐσία, τοῦτο πρὸς πίστιν ἀλήθεια.

[6] Ibid., 29d,1162. τὸν εἰκότα μῦθον.

of natural things) do not rise to the knowledge we possess of the eternal, immaterial, and unchangeable realm of patterns—the Forms. The so-called mathematical *precision-problem*[7] is no surprise to Plato; indeed, it is the imperfection in and changeability of material reality—including the dynamic quantitative structures of *things*—that renders our *understanding* of such things imperfect and changeable.[8] On the other hand, perfect mathematical knowledge must be *about* perfect mathematical realities.

A similar account of the objects of mathematics and mathematics itself is found in several sections of the *Republic*. The general structure of the dialog lends itself to addressing the study of mathematics as a part of political life, as well as questions regarding the nature of mathematical objects and whether, and how, we know them.

As with many of the dialogues, the *Republic* begins with a central question, in this case, 'what is justice?'[9] In the course of the discussion that ensues, Socrates proposes that the rulers of the ideal (i.e., most just) city

[7] The precision-problem arises from the obvious difference between physical instances of things like lines and surfaces and their abstract mathematical counterparts. As Jonathan Lear points out, it is generally thought that Plato's mathematicals are not physically instantiated and that mathematicals do not physically have the character the mathematician proves of the objects of math. By this account the precision problem would *not* be a problem for mathematical platonists. On the other hand, as to whether Plato himself would or would not have been troubled by it, see especially Part Two of Pritchard's *Plato's Philosophy of Mathematics*. Paul Pritchard, *Plato's Philosophy of Mathematics* (Sankt Augustin Germany: Academia Verlaag, 1995).

[8] Cf. Thomas Aquinas, S*umma Theologiae* (Ottawa: Commissio Piana, 1953), I 84 1, hereafter *Summa Theologiae*. Aquinas argues for an epistemology midway between ancient materialism and platonism, pointing out that each view makes the same mistake, that of ascribing an object's properties to our awareness of that object. His is an application of the famous *dictum* "whatever is received is received according to the mode of the receiver" to our acts of knowing.

[9] Cephalus introduces the topic of justice early in the dialogue. Trans. Plato, "Republic", trans. Paul Shorey in *The Collected Dialogues of Plato*, eds. Edith Hamilton and Huntington Cairns (New Jersey: Princeton University Press, 1978), 331a, 580. "For a beautiful saying it is, Socrates, of the poet that when a man lives out his days in justice and piety 'sweet companion with him, to cheer his heart and nurse his old age, accompanies Hope, who chiefly rules the changeful mind of mortals'. That is a fine saying and an admirable." χαριέντως γάρ τοι, ὦ Σώκρατες, τοῦτ' ἐκεῖνος εἶπεν, ὅτι ὃς ἂν δικαίως καὶ ὁσίως τὸν βίον διαγάγῃ,

γλυκεῖά οἱ καρδίαν
ἀτάλλοισα γηροτρόφος συναορεῖ
ἐλπὶς ἃ μάλιστα θνατῶν πολύστροφον
γνώμαν κυβερνᾷ. [Pindar fr. 214]
εὖ οὖν λέγει θαυμαστῶς ὡς σφόδρα.

must be philosophers, whose education must be the best.[10] But what sort of education will this be? Philosophical, of course—which is to say, having *the good* as its object.[11] It is at this point we hear how knowledge of the good is to be obtained, the course plotted by Socrates by way of famous analogies: the sun, the divided line, and the cave.[12] As to details, he sketches out "the study that would draw the soul away from the world of becoming to the world of being"[13] which includes, in turn, music, gymnastics, and the arts,[14] pure arithmetic and geometry,[15] astronomy,[16] harmonics,[17] and, finally, dialectics.[18]

Politically, then, the realm of the mathematicals—rather our study of them—supplies us with an insight into the unchangeable, the eternal, the world of *what is* as opposed to the world of *what is coming to be*. One can keep true justice in one's sight only when once accustomed to discerning and working with eternal realities. In this way mathematics is a path to recognizing that *real* order among us whereby human thriving is achieved.

Plato's overall point seems clear—to rule justly, rulers must have a clear notion of the nature of justice itself. It is no surprise that realist

[10] Ibid., 502c–d, 738. "This difficulty disposed of, we have next to speak of what remains, in what way, namely, and as a result of what studies and pursuits, these preservers of the constitution will form a part of our state, and at what ages they will severally take up each study." οὐκοῦν ἐπειδὴ τοῦτο μόγις τέλος ἔσχεν, τὰ ἐπίλοιπα δὴ μετὰ τοῦτο λεκτέον, τίνα τρόπον ἡμῖν καὶ ἐκ τίνων μαθημάτων τε καὶ ἐπιτηδευμάτων οἱ σωτῆρες ἐνέσονται τῆς πολιτείας, καὶ κατὰ ποίας ἡλικίας ἕκαστοι ἑκάστων ἁπτόμενοι;

[11] Ibid., 505a, 740. "For you have often heard that the greatest thing to learn is the idea of good by reference to which just things and all the rest become useful and beneficial." ἐπεὶ ὅτι γε ἡ τοῦ ἀγαθοῦ ἰδέα μέγιστον μάθημα, πολλάκις ἀκήκοας, ᾗ δὴ καὶ δίκαια καὶ τἆλλα προσχρησάμενα χρήσιμα καὶ ὠφέλιμα γίγνεται.

[12] Ibid., 507c-509d, 742–5; 509d–511e, 745–7; and 514a–517c, 747–50.

[13] Ibid., 521d, 753.

[14] Ibid., 522b, 754.

[15] Socrates relies upon the distinction between pure mathematics and applied mathematics (after all, a soldier also needs 'to reckon and number'). In their search for a knowledge of the true, philosopher-kings would look beyond the world of the changeable and perceptible to the eternal. In one place Socrates says "…[reckoning and the science of arithmetic] would be among the studies that we are seeking. For a soldier must learn them in order to marshal [μαθεῖν, mathein] his troops, and a philosopher because he must rise out of the region of generation and lay hold on essence or he can never become a true reckoner." Ibid., 525b, 757. "πολεμικῷ μὲν γὰρ διὰ τὰς τάξεις ἀναγκαῖον μαθεῖν ταῦτα, φιλοσόφῳ δὲ διὰ τὸ τῆς οὐσίας ἅπτεόν εἶναι γενέσεως ἐξαναδύντι, ἢ μηδέποτε λογιστικῷ γενέσθαι."

[16] Ibid., 528d-530d, 761–2.

[17] Ibid., 530d-531c, 763.

[18] Ibid., 531c-535a, 763–7.

mathematicians identify their own views with Plato—perhaps not Platonism (i.e., with the very account of the Forms and Mathematicals one finds here) but with platonism generally, i.e., the view that the sort of precise and unchanging knowledge which we hope for or obtain in mathematics requires that its objects be separate, abstract, unchangeable. After all, if beauty is unchanging, it is not an individual face which *constitutes* beauty—one only *borrows* beauty for a brief time.[19] All the more so, then, for mathematics, since no physical instance of the number two (e.g., two eyes) has that permanent, unchangeable character which we expect the subject about which we predicate 'even number' always to have. There is good reason, then, to classify Plato's account of mathematics as a *realism*.

The Justification for, and Place of, Plato's Mathematicals

Delving more carefully into one of the aforementioned analogies—the divided line—we can further specify the kind of existence Plato ascribes to these abstract entities.[20]

Aristotle supplies a helpful indication of why Plato's mathematicals, though abstract and real, must differ from the abstract forms (like the Good) to which he makes repeated references.

> [Plato] says the mathematicals exist mid-way between sensible things and the Forms, differing from sensible things in being eternal and immovable and from the Forms in there being more than one [mathematical] the same, whereas each Form is itself only one.[21]

[19] The need for the abstract forms of Beauty, Truth, and Goodness, is found in many of Plato's dialogues. This particular example, and discussion, occur in the *Cratylus*.

[20] We emphasize once again that historical developments in how mathematics is regarded by philosophers and mathematicians themselves was predicated, in part, upon a certain way of understanding Plato. Plato scholars are deeply divided as to whether Plato himself ever ascribed to mathematicals the intermediate status which forms the basis of this section. For a sense of the controversy and the range of views see the summaries from John A. Brentlinger, "The Divided Line and Plato's 'Theory of Intermediates'," *Phronesis* 8 no. 2 (1963): 146–66—whose *status quaestionis* goes back to the latter part of the nineteenth century—J. M. Rist, "Equals and Intermediates in Plato," *Phronesis* 9 no. 1 (1964): 27–37, and Julia Annas, "On the 'Intermediates'," *Archiv für Geschichte der Philosophie* 57 no. 2 (1975): 146–66.

[21] Aristotle, *Metaphysica* (Oxford: Oxford University Press, 1957), I 6987b14–18, hereafter *Metaphysica*. ἔτι δὲ παρὰ τὰ αἰσθητὰ καὶ τὰ εἴδη τὰ μαθηματικὰ τῶν πραγμάτων εἶναί φησι μεταξύ, διαφέροντα τῶν μὲν αἰσθητῶν τῷ ἀΐδια καὶ ἀκίνητα εἶναι, τῶν δ᾽ εἰδῶν τῷ τὰ μὲν πόλλ᾽ ἄττα ὅμοια εἶναι τὸ δὲ εἶδος αὐτὸ ἓν ἕκαστον μόνον.

The genesis of Plato's Forms is epistemological. It is significant that the accounts from both the *Timaeus* and the *Republic* are rooted in questions about knowing—knowing for a political end, in fact, but still knowing. What is it about our knowing that prompted Plato to posit separate abstract substances? We believe the *Cratylus* supplies a clear reason. Near the end of this dialogue about the origin and usage of words, Socrates raises a question about the *basis* for our naming things.

> There is a matter, master Cratylus, about which I often dream, and should like to ask your opinion: Tell me whether there is, or is not, any absolute beauty or good, or any other absolute existence?[22]

Surely the beautiful-itself and the good-itself are unchangeable, though beautiful and good things among us are not. If beauty were in a face, what of our knowledge of beauty when that face ceases to be so?

> Then how can that be a real thing which is never in the same state? for obviously things which are the same cannot change while they remain the same; and if they are always in the same state and the same, then, without losing their original form, they can never change or be moved.[23]

The beautiful-itself and the good-itself are, then, eternal and unchangeable, else we could not explain our knowledge of them.

> Nor yet can they be known by anyone; for at the moment that the observer approaches, then they become other and of another nature, so that you cannot get any further in knowing their nature and state, for you cannot know that which has no state.[24]

This interplay between Plato's epistemology and his ontology is ubiquitous. To return to the analogy of the divided line, why should mathematicals be regarded differently from the Forms themselves if, as Aristotle

[22] Plato, "Cratylus", trans. Benjamin Jowett in *The Collected Dialogues of Plato*, eds. Edith Hamilton and Huntington Cairns (New Jersey: Princeton University Press, 1978), 439c–d, 473. σκέψαι γάρ, ὦ θαυμάσιε Κρατύλε, ὃ ἔγωγε πολλάκις ὀνειρώττω. Πότερον φῶμέν τι εἶναι αὐτὸ καλὸν καὶ ἀγαθὸν καὶ ἓν ἕκαστον τῶν ὄντων οὕτω, ἢ μή;

[23] Ibid.

[24] Ibid., 440a, 473. ἀλλὰ μὴν οὐδ᾽ ἂν γνωσθείη γε ὑπ᾽ οὐδενός. Ἅμα γὰρ ἂν ἐπιόντος τοῦ γνωσομένου ἄλλο καὶ ἀλλοῖον γίγνοιτο, ὥστε οὐκ ἂν γνωσθείη ἔτι ὁποῖόν γέ τί ἐστιν ἢ πῶς ἔχον: γνῶσις δὲ δήπου οὐδεμία γιγνώσκει ὃ γιγνώσκει μηδαμῶς ἔχον.

contends, Plato does so regard them? Because mathematics proceeds differently from other areas of study.

Julia Annas references this difference as a response to the so-called *uniqueness problem*.

> Intermediates are thus required for the mathematical sciences in order to preserve the apparent sense of mathematical statements without having to admit that these statements are about either Forms or physical objects. … The intermediates are the answer to a specific problem which arises only with the mathematical Forms. I shall refer to this as 'the Uniqueness Problem.'[25]

As are others who investigate the status of mathematicals in Plato and Aristotle, Annas is understandably puzzled by Aristotle's characterization of Plato's view via *Metaphysics* I 6, given that Plato does not seem to have explicitly described them in these terms. So why would he do so? The argument is made that Aristotle is either drawing out what is implicitly contained in Plato's literal accounts or is developing the platonic view along these necessary lines. Thus Brentlinger:

> It is hardly necessary to go into the intricacies of an interpretation of Aristotle's interpretation of Plato in order to accept these distinctions as true; the overwhelming amount of evidence in the *Metaphysics* and the almost universal agreement of scholars on the matter, makes the existence of a Platonic theory of intermediates, along with ideas and sensibles, highly probable.[26]

The received wisdom up through the larger part of the twentieth-century—until mathematicians themselves began to distinguish mathematical platonism from historical Platonism, perhaps—is that Plato's account of mathematics requires that mathematicals *also* be intermediates between the Forms and sensible things. Brentlinger supplies the desired connection, notably, in relation to how mathematics *is done*:

> Cook Wilson has shown, in my opinion conclusively, that the view of the ideal numbers as ἀσύμβλητοι ἀριθμοί, and the theory of intermediates, of

[25] Annas, "On the 'intermediates'," 151.

[26] Brentlinger, "The Divided Line and Plato's 'Theory of Intermediates'," 146. As to the evidence, Brentlinger references David Ross, *Plato's Theory of Ideas* (Oxford: Oxford University Press, 1953), 151–3.

> which Aristotle speaks, are required by Plato's theory of ideas from the very beginning. According to his discussion, which is adequate for my purpose, the ideal numbers and figures are not quantities at all, but are rather qualities, such as square-ness and triangularity, or twoness and threeness. Aristotle calls these entities ἀσύμβλητοι, 'incomparable,' or in the case of numbers 'inaddible'; they are so, of course, because as squareness, or as twoness, they have no determinate properties that are usable in mathematical studies. One does not inscribe triangularity inside circularity: the ideal circle has no area, no diameter, no circumference, no center; the ideal square is not a four-sided entity with an incommensurate diagonal.[27]

So, too, Annas:

> There is, however, a special problem with numbers which does not arise in the case of the other Forms. In everyday life we talk of adding numbers, and of performing other repeatable operations on them. This is clear from the simplest arithmetical statement, like '2+2=4',' or indeed as soon as we begin to talk about 'twice two.' What we say seems to have a clear sense and a satisfactory use. But what are we talking about when we use such a statement? Not groups of physical objects, for the truth of such statements does not depend on the observable behaviour of physical groups. But surely not the Form number Two either, for this is unique, and it can make no sense to talk of adding it to itself. Similarly it makes no sense to talk of Twoness being part of Threeness, or of getting Twoness by adding two Onenesses. Surely, however, '2+2=4' must be about numbers? So it appears that it must be about a third kind of number, distinct from numbered groups and also distinct from Forms. This will be mathematical number, the number we do mathematics with.[28]

This distinction, which seems to be brought to Plato's account of mathematics despite Plato himself, as Cook Wilson says, "…embodies a truth which depends on nothing peculiar to Platonism."[29] Mathematics itself—the science and discipline practiced by such persons—seems to have led to an observation regarding Plato's own account of it which he may not have explicitly made, though which seems presupposed to *any* workable account.

[27] Brentlinger 1963, p. 159. He references J. Cook Wilson, "On the Platonist Doctrine of the ἀσύμβλητοι ἀριθμοί," *Classical Review* 18, no. 5 (1904): 247–60.

[28] Annas, "On the 'intermediates'," 150.

[29] Wilson, "On the Platonist Doctrine of the ἀσύμβλητοι ἀριθμοί," 250.

Why Is Platonism Attractive to Some Philosophies of Mathematics?

If we are now content to call platonism that account of mathematics which insists upon abstract, real objects of which there can be more than one the same, it is fairly easy to see why working mathematicians, as well as some philosophers of mathematics, would espouse such a view.

In the latter part of the nineteenth century, Gottlob Frege wrote his highly influential[30] *Foundations of Arithmetic*, arguing for mathematical realism in the face of those who would explain mathematical claims, such as $2 \times 2 = 4$, as mere by-products of the human brain rather than eternally true descriptions of objects existing in a separate realm.[31] As he says in his *Introduction*:

> A proposition may be thought, and again it may be true; let us never confuse these two things. We must remind ourselves, it seems, that a proposition no more ceases to be true when I cease to think of it than the sun ceases to exist when I shut my eyes. Otherwise, in proving Pythagoras' theorem we should be reduced to allowing for the phosphorous content of the human brain; and astronomers would hesitate to draw any conclusions about the distant past, for fear of being charged with anachronism, with reckoning twice two as four regardless of the fact that our idea of number is a product of evolution and has a history behind it. It might be doubted whether by that time it had progressed so far. How could they profess to know that the proposition $2 \times 2 = 4$ was already in existence in that remote epoch? Might not the creatures then extant have held the proposition $2 \times 2 = 5$, from which the proposition $2 \times 2 = 4$ was only evolved later through a process of natural

[30] That is, ultimately. It took the efforts of Bertrand Russell, Guiseppe Peano, and others to drive home the importance of Frege's works.

[31] Cf. Gottlob Frege, *The Foundations of Arithmetic,* trans, J. L. Austin (New York: Harper, 1960). Frege references this realm in a later work, *Thoughts.* "So the result seems to be: thoughts are neither things of the outer world nor ideas. A third realm must be recognized. What belongs to this corresponds with ideas, in that it cannot be perceived by the senses, but with things, in that it needs no bearer to the contents of whose consciousness to belong. Thus the thought, for example, which we expressed in the Pythagorean theorem is timelessly true, true independently of whether anyone takes it to be true. It needs no bearer. It is not true for the first time when it is discovered, but is like a planet which, already before anyone has seen it, has been in interaction with other planets." Gottlob Frege, "Thoughts," trans. P. T. Geach, *Mind* 65 no. 259 (1956): 302.

> selection in the struggle for existence? Why, it might even be that 2 × 2 = 4 itself is destined in the same way to develop into 2 × 2 = 3![32]

Frege was struck, not by the complexity of the mathematics of his day, but by the simplest claims—foundational claims. As he says, if we cannot even explain what a number is, what we mean when we say 1 + 1 = 2:

> …[q]uestions like these catch even mathematicians for that matter, or most of them, unprepared with any satisfactory answer. Yet is it not a scandal that our science should be so unclear about the first and foremost among its objects, and one which is apparently so simple? Small hope, then, that we shall be able to say what number is. If a concept fundamental to a mighty science gives difficulties, then it is surely an imperative task to investigate it more closely until those difficulties are overcome…[33]

As an objective basis for the timeless truths of mathematics, as Frege sees it, platonism supplies the only alternative to both the empirical (and so revisable) sciences and the subjectivity consequent upon of *psychological* explanations for mathematics and its claims. The mathematicals, after all, are *objects*:

> Now, in the case of a it is quite right to decline to answer [what is a?]: a does not mean some one definite number which can be specified, but serves to express the generality of general propositions. If, in $a + a - a = a$, we put for a some number, any we please but the same throughout, we always get a true identity. With one, however, the position is essentially different. Can we, in the identity 1 + 1 = 2, put for 1 in both places some one and the same object, say the Moon? On the contrary, it looks as though, whatever we put for the first 1, we must put something different for the second.[34]

Georg Cantor, like Frege,[35] was also concerned both with mathematics and the philosophizing needed for its justification. We shall deal with Cantor other more-contemporary mathematical realists in Part Three.

[32] Ibid., xviii–xix.
[33] Ibid., xiv.
[34] Ibid., xiii–xiv.
[35] For Frege on the need for philosophy in mathematics, see ibid., xvii.

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CHAPTER 3

Aristotle on the Objects of Mathematics

Where Do the Mathematicals Exist?

We have briefly presented the sort of account of the mathematicals and mathematics with which Aristotle would have been familiar by way of Plato, as well as its importance in the development of mathematical thought to this day. As we approach the distinctive form of explanation Thomas Aquinas supplies, it is necessary to turn to Aristotle himself—both as to what he supports and as to those aspects of Plato's thought he contests—if we wish to have a solid understanding of the medieval thinker.

We have divided our treatment of Aristotle into four chapters, the present one concerned with the mathematicals, the very objects under consideration by the mathematician, a second dealing with the nature of mathematical claims, both individually and as a united body of knowledge, a third touching upon intellectual acts which serve most sharply to distinguish Aristotle from Plato and which connect mathematical object to systematic mathematical thought,[1] and, lastly, how one must characterize Aristotelian mathematical realism in its distinction from other such realisms.

From the time of Plato and Aristotle through Frege, Cantor, and even Quine, the existence of mathematical objects has been a matter of speculation and philosophical debate. It has all the earmarks of a perennial

[1] Sometimes called the access problem among mathematical platonists.

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question. Mathematics is an attractive target for the skeptically minded since it has the reputation of supplying universal and necessary claims which are beyond question,[2] while there will always be those who insist—and perhaps some who over-insist—upon the timelessness, and even god-like character, of the mathematical realm.[3]

Now, mathematics is *about something*, at least, so math as a system would require that we first know its principles, for example, 'when to equals are added equals the results are equal,' and the like, as well as the meaning of terms such as 'prime,' 'square,' and so on. We would finally (which is really to say *firstly*) need to know both *what* its objects are and *that* such things exist.

But *where*? Could they possibly exist in physical bodies, like other accidents which substances have? Following Aristotle's account, numbers, triangles, circles, and suchlike are found in the category of quantity.[4] But quantities are accidents of substances, and so mathematicals must also be said to exist *in* substances.

But how is this possible? Suppose a dozen cows. By this account the number twelve *belongs* to them. Not to *one* of the cows as, for example, its color belongs: for that does not depend upon the other cows in the group. While the others might leave, a brown cow would remain brown—not so with twelve. Moreover, it is simply false to say that the number twelve belongs to a single substance. One is not twelve—beyond false, that is demonstrably a contradiction. It remains that twelve belongs to *all* the cows assuming (as Aristotle does, it seems) it does belong. Also to *each*? If not, in what substance does *twelve* inhere?

Now suppose half of the cows are standing to the north of a tree. We have argued that *twelve* belongs to all the cows: but those north of the tree are six. By equivalent reasoning, *six* belong to those. Thus, the six are both *six* and *twelve*.

Of course, these claims presuppose relative differences. *Twelve* inheres in the larger group and *six* in the smaller—nothing difficult in this. What number inheres in all possible groups of cows? As many as there are cows, of course. Yet how could cows possess so many real and distinct attributes? If *twelve* belongs to the larger group and *six* to the smaller and so on, and we recognize that these numbers change only according to our perspective, it seems much more reasonable to say that these numbers are due to

[2] *Pace* Sextus Empiricus and René Descartes.

[3] Cf. what we said of Cantor, above.

[4] See Aristotle, *Categoriae* (Oxford: Oxford University Press, 1980) 6 4b23–25, hereafter Categoriae, and *Metaphysica* V 71017a23–28.

thought and not to reality. Far more reasonable to conclude that each animal is merely one and groups are numerable, in virtue of our mind's ability to understand and take note of yet one more unit.

It is entirely *unclear*, then, where mathematicals exist, if they exist in physical substances.

On the other hand, Aristotle (and Thomas Aquinas, for that matter) insists, and quite plainly, that the objects of mathematics exist outside the mind. Quantity is a logical and metaphysical category, divided into the continuous and the discrete, the continuous in one, two, and three dimensions, and the discrete as number or speech. But the other nine categories (besides substance) name attributes which are *present in* things (namely, in physical substances).

Following Aristotle, Thomas Aquinas makes the point clear in his *Commentary on the Metaphysics*:

> …there are even principles, elements, and causes of mathematical things, for example, of figure and number and others of this sort which the mathematician considers. In general, every intellectual science, howsoever much it has a share in intellect [has principles, causes, and elements] … even [sciences which are about] sensible things to the extent that there is a science of them, such as mathematics and natural science.[5]

Moreover, however strange it may sound to hear Aquinas claim that mathematics studies *sensible things*, he and Aristotle both maintain that the objects of mathematics are in fact *the same as* those of natural philosophy. So clear is this to them, that it raises a question whether the two sciences *even differ*. As Aristotle says in *Physics*: "We must next ask how the mathematician differs from the philosopher of nature, for physical bodies have surfaces and volumes and lines and points, about which the mathematician is concerned."[6]

[5] Thomas Aquinas, *In Duodecim Libros Metaphysicorum Aristotelis Expositio* (Rome: Marietti Editori Ltd., 1964), VI 1 1145, hereafter *In Metaphysicorum*. "Et similiter etiam mathematicorum sunt principia et elementa et causae, ut figurae et numeri et aliarum huiusmodi quae perquirit mathematicus. Et universaliter omnis scientia intellectualis qualitercumque participet intellectum … et etiam sensibilia prout de his est scientia, sicut in mathematica et in naturali."

[6] Aristotle, *Physica* (Oxford: Oxford University Press, 1982), II 2 193b23–24, hereafter *Physica*. μετὰ τοῦτο θεωρητέον τίνι διαφέρει ὁ μαθηματικὸς τοῦ φυσικοῦ (καὶ γὰρ ἐπίπεδα καὶ στερεὰ ἔχει τὰ φυσικὰ σώματα καὶ μήκη καὶ στιγμάς, περὶ ὧν σκοπεῖ ὁ μαθηματικός).

This point is driven home by way of Aquinas' comments upon the passage:

> Any sciences considering the same subjects are either the same, or one is a part of the other: but the mathematical philosopher considers points, lines, surfaces, and bodies, and so does the natural philosopher (since he proves that natural bodies have planes, that is, surfaces, and volumes, that is, solids, and lengths and points; and it is necessary that the natural philosopher consider all the things that are in natural bodies); therefore it seems that either natural science and mathematics are the same, or that one is a part of the other.[7]

What an intellectual system is *about* distinguishes it from other such systems. Mathematicians are not natural philosophers or metaphysicians—they are *about* numbers, solids, lines, points, and surfaces—it is about these things that they prove certain properties. The problem remains, since philosophers of nature and even metaphysicians are about these things, too—evidence this book, for example.

Aristotle solves the difficulty by noting that *how* one systematizes one's thought makes all the difference: "geometry considers the physical line, but not as physical, whereas optics [considers] the mathematical line, yet not as mathematical but as physical."[8]

The mathematical sciences are about mathematical things, yes, but only considered apart from certain aspects of the substances in which they are

[7] Thomas Aquinas, *In Octo Libros Physicorum Expositio* (Rome: Marietti Editori Ltd., 1965), II 3 158, hereafter *In Physicorum*. "Quaecumque scientiae considerant eadem subiecta, vel sunt eaedem, vel una est pars alterius; sed mathematicus philosophicus considerat de punctis, lineis et superficiebus et corporibus, et similiter naturalis (quod probat ex hoc quod corpora naturalia habent plana, idest superficies, et firma, idest soliditates, et longitudines et puncta; oportet autem quod naturalis consideret de omnibus quae insunt corporibus naturalibus); ergo videtur quod scientia naturalis et mathematica vel sint eadem, vel una sit pars alterius."

[8] *Physica*, II.2.194a9–12. ἡ μὲν γὰρ γεωμετρία περὶ γραμμῆς φυσικῆς σκοπεῖ, ἀλλ᾽ οὐχ ᾗ φυσική, ἡ δ᾽ ὀπτικὴ μαθηματικὴν μὲν γραμμήν, ἀλλ᾽ οὐχ ᾗ μαθηματικὴ ἀλλ᾽ ᾗ φυσική. So, too, Thomas Aquinas: "Geometry considers the line which has being in sensible matter, which is the natural line: nevertheless, it does not consider it insofar as it is in sensible matter, according as it is natural, but abstractly, as was said." *In Physicorum*, II.3.164. "Nam geometria considerat quidem de linea quae habet esse in materia sensibili, quae est linea naturalis: non tamen considerat de ea inquantum est in materia sensibili, secundum quod est naturalis, sed abstracte, ut dictum est."

found.[9] In traditional terms, mathematics is about things abstracted from physical matter.[10] Even so, we are driven to the same conclusion: what is abstracted from physical matter must exist *in* physical matter. Physical things *have* quantities. Those quantities, once we ignore their *physicality*, is what mathematics is about.

Our first real problem, then, is that, on the one hand, following the instance of counting cows, mathematicals cannot exist in physical substances, for we could not say *where* they might be found other than in the mind of the one counting them. On the other hand, Aristotle (and Thomas Aquinas) maintains that that is *precisely where* such things must be found.[11]

Aristotle divides 'being' in many ways, one of which is into *essential* and *accidental*[12]:

> Being is said to be accidental or essential. Accidental, as when we say that a just man is musical or that a man is musical or that a musician is a man, just as we might say that a musician builds because a builder happens (συμβέβηκε) to be a musician or a musician a builder. For 'this is that' here means 'this happens to be that.'[13]

Accidents in this sense are complex—not quantities or qualities or somesuch, but two or more things united in a certain *manner*, like the predicable relation which is also called an *accident*. Musicality and the art of building are not mutually essential: not every builder is also a musician, nor vice versa. *Musical builder* is an *accident*, or, perhaps better, is *accidentally* a being. Unlike *categorical* accidents, such accidents are not opposed to substance, rather, they are opposed to what is *essential*:

[9] Note that we are still operating under the assumption that Aristotelian mathematical realism sees the mathematicals as attributes of physical substances.

[10] As we will see, this sort of abstraction differs from the sort that drove Plato to posit the Forms and the Intermediates.

[11] As seems reasonable. After all, how can we get multiplicity of any sort without number?

[12] *Per se* and *per accidens* are Latin transliterations of καθ᾽ αὑτό and κατὰ συμβεβηκός. Aquinas reminds us that categorical accidents are not beings *per accidens*.

[13] *Metaphysica*, V 7 1017a7–13. Τὸ ὂν λέγεται τὸ μὲν κατὰ συμβεβηκὸς τὸ δὲ καθ᾽ αὑτό, κατὰ συμβεβηκὸς μέν, οἷον τὸν δίκαιον μουσικὸν εἶναι φαμεν καὶ τὸν ἄνθρωπον μουσικὸν καὶ τὸν μουσικὸν ἄνθρωπον, παραπλησίως λέγοντες ὡσπερεὶ τὸν μουσικὸν οἰκοδομεῖν ὅτι συμβέβηκε τῷ οἰκοδόμῳ μουσικῷ εἶναι ἢ τῷ μουσικῷ οἰκοδόμῳ (τὸ γὰρ τόδε εἶναι τόδε σημαίνει τὸ συμβεβηκέναι τῷδε τόδε).

> What is said to be *essentially* signifies just as the figures of the categories do: in howsoever many ways these are said, in so many ways does being signify. Now since, of these categories, some signify what it is, others of what sort, others how much, others in relation to what, others what it does or suffers, others where, and others when, to each of these there [corresponds] a being which it signifies.[14]

Essential being equates to a definite nature—be it a substance or an accident.[15]

What of the mathematicals? Clearly, for Aristotle, they must be beings *per se*—they fall into quantity, one of the ten categories, each of which signifies essential being of one sort or another. Importantly, they are not *accidents* (in this second sense), as if we were to say that each is a composite whose members may or may not be present, or whose members are merely accidentally related, the *musical builders* of the mathematical world. Each mathematical is a complete nature.

We arrive at the same: the Aristotelian answer to the question "where do mathematicals exist?" is "in physical substances."[16]

Earlier, we noted that the number of cows in any given group is reckoned arbitrarily, subject to our frame of reference. Two possibilities then arose: (1) *twelve* is in all or (2) *twelve* is in each. Yet neither case implied a *real twelve*.

Applying the being *per se* and being *per accidens* distinction, *twelve* is not in all the cows *essentially*. Categorical accidents (like the mathematicals) belong to substances, not to groups. Even supposing that *twelve* is still in the cows, but *mediately*—through being in the group, which is a group of cows—it remains that a group is not itself a being *per se* (it is like *musical builder,* not like *musical* or *builder*). *Twelve*, then, is not in all the cows *per se*. Nor, obviously, is it in each of them *per se*.

Let us restrict our discussion to two cows but consider only their legs. The statement 'the cows' legs are eight in number' is of the same form as 'the cows are twelve in number' and would give rise to similar difficulties. Yet let us compare our new claim to another, 'the spiders' legs are eight in

[14] Ibid., V 7 1017a23–28. καθ᾽ αὑτὰ δὲ εἶναι λέγεται ὅσαπερ σημαίνει τὰ σχήματα τῆς κατηγορίας· ὁσαχῶς γὰρ λέγεται, τοσαυταχῶς τὸ εἶναι σημαίνει. ἐπεὶ οὖν τῶν κατηγορουμένων τὰ μὲν τί ἐστι σημαίνει, τὰ δὲ ποιόν, τὰ δὲ ποσόν, τὰ δὲ πρός τι, τὰ δὲ ποιεῖν ἢ πάσχειν, τὰ δὲ πού, τὰ δὲ ποτέ, ἑκάστῳ τούτων τὸ εἶναι ταὐτὸ σημαίνει.

[15] Bearing in mind that accidents are essentially dependent upon substances.

[16] Aristotle points out that one of the senses of *in* is *as a form is in matter*. See *Physica*, IV 3.

number.' This last is ambiguous. We could mean 'the legs of all the spiders under consideration *collectively* number eight' (such that some have fewer than eight) or 'the legs of *each* of the spiders under consideration number eight.'[17] What of the second form, that *each* spider has eight legs? Is *eight* in the legs of a spider—in the legs of *each* spider—differently than in the legs of two cows?

If we allow ourselves to argue from the *form* of these expressions, we might expect to see the same difficulties arise here as well. As with the cows and their legs we might say 'each spider's legs are both eight and four, in that they are eight on the spider itself, yet four on the left side,' and be led to the same absurdities as before. Yet here there is a *preferred frame of reference*, so to speak, for we may say that the spider *naturally* has eight legs, meaning thereby that there would be something *less* in a spider with fewer legs (and not *less* as *fewer in number*) or in a spider with a greater number of legs. Furthermore, here we can *say* where (in which physical substance) it is: the individual, physical spider.

Now, perhaps we are speaking of an arbitrary—albeit convenient—whole when we speak of a spider. After all, we are not numbering the spider but its legs. Enter the same difficulties. *Eight* is not in any leg, nor in all the legs for, in removing a leg (changing its place) one reduces the number to seven. Substituting spider legs for cows achieves nothing.

Or does it? There is a difference in the two cases still. On the one hand, a group of cows has no essential *unity*—any collective unity was *given* to them by us, which is accidental to the cows themselves. Is this difference sufficient to show that eight *really is* here but not elsewhere?

One of the ancient philosophers, Democritus, approximates a distinction between the essential and accidental which is similar to Aristotle's own. He maintains that the first principles of all natural things are atoms and the void, where *atom* equates to *indivisible bodies*. Aristotle develops Democritus' principle along these lines:

> If substance is one, it will not be constituted of substances in this way [as actually present in it], as Democritus rightly says. For it is impossible (he

[17] 'The cows' legs are eight in number' is of the first form. Two cows have eight legs only collectively.

says) for one to come from two or for two to come from one: for he makes his atoms substances.[18]

If atoms and nothingness are all there is, a compound body is not a real thing. This is what Democritus means when he says "it is impossible for one to come from two." No compound body is a real thing, for no compound is an atom, (it is, rather, a collection of atoms,) and *only* atoms are real things. As Democritus has it, there is a real thing (like Aristotle's essential being), and what is not really a thing at all, a collection (like Aristotle's accidental being). A collection of atoms, though many in number, is not one thing. In short, *wherever there is number, there is no unity.* Taking it one step further, number itself is merely an accidental collection of units—not a real unity and not a real thing.

Aristotle's point in introducing this account by Democritus is to criticize it. Of course, as we have seen, the Aristotelian mathematicals exist and exist *per se*—which is some support for their having their own *unity* as well. It is obvious that numbers, at least, are some sort of plural—arguably, mathematics itself is *about* plurality—does this require that they are therefore not one?

Apparently, we can only measure and count things having *some* unity. Each of these activities, once begun, can stop only upon our having reached a (non-arbitrary) limit, and to limit is, in some way, to make things one. Inevitably, we group objects we intend to count if they are not already so grouped. This is analogously true for measures. Apparently, *unity and diversity* are presupposed in what we count and measure. *Essential* unity?

As with *being*, *unity* is necessarily of many sorts: for unity *means* undivided being.[19] No wonder that Aristotle includes both unity *per se* and unity *per accidens* among its various senses.[20] If the mathematicals have unity of some sort, of which sort?

We have already noted that a group of cows is one accidentally. Their unity does not reside in them but is wholly external, in the mind of the

[18] *Metaphysica*, VII 13 1039a7–11. εἰ ἡ οὐσία ἕν, οὐκ ἔσται ἐξ οὐσιῶν ἐνυπαρχουσῶν καὶ κατὰ τοῦτον τὸν τρόπον, ὃν λέγει Δημόκριτος ὀρθῶς: ἀδύνατον γὰρ εἶναί φησιν ἐκ δύο ἓν ἢ ἐξ ἑνὸς δύο γενέσθαι: τὰ γὰρ μεγέθη τὰ ἄτομα τὰς οὐσίας ποιεῖ.

[19] See *Ibid*., X 3. Aristotle has already distinguished this transcendental sense of *one* from the numerical unit in *Ibid*., V 6.

[20] Ibid., V 6.

one counting them. A spider, however, is not accidentally but essentially one. It is a living thing, unified by its soul, the form of its substance.[21]

But what do we mean by saying that the spider is essentially one? However true it might be to say that it is one thing by nature, the notion of *essentially one* remains ambiguous. One in number? If so, it is many, too, for numerically it has many parts and each part is itself one part, and a whole is equal to the sum of its parts. Whatever incremental progress we make in distinguishing the eight legs of two cows from the eight legs of a spider, many puzzles remain.[22]

As to what *one* can mean, Aristotle describes things which have unity *per accidens* in much the same way as those which have being *per accidens*—his examples are virtually identical.

> 'One' is said accidentally or essentially. For example, 'Coriscus' and 'the musical' and 'musical Coriscus' are accidentally [one] … It is the same whether the accident is said of a genus or of some universal name, for example, that 'man' and 'musical man' are the same: for this is either because 'the musical' happens to belong to 'man', which is one substance, or because both happen to belong to some individual, as Coriscus (except that they do not both belong [to him] in the same way, but perhaps one as a genus and in his substance and the other as a possession or characteristic of his substance).[23]

Musical man both is, and is one, *per accidens,* which makes sense. After all, everything accidental is complex, though the converse is not true, since what is one *per se* may be so in virtue of a kind of continuity within its parts. Aristotle continues:

> Of things that are called one essentially, some are said to be [such] by continuity, as a bundle by a bond and pieces of wood by glue; and a line, even

[21] Substances have unity *per se*. Again, see *Ibid*., V 6.

[22] We recall Socrates' declaration, upon recounting his own like considerations of unity and plurality, that he was wholly incapable of such investigations.

[23] *Metaphysica*, V 6 1015b16–18, 29–34. ἓν λέγεται τὸ μὲν κατὰ συμβεβηκὸς τὸ δὲ καθ᾽ αὑτό, κατὰ συμβεβηκὸς μὲν οἷον Κορίσκος καὶ τὸ μουσικόν, καὶ Κορίσκος μουσικός… ὡσαύτως δὲ κἂν ἐπὶ γένους κἂν ἐπὶ τῶν καθόλου τινὸς ὀνομάτων λέγηται τὸ συμβεβηκός, οἷον ὅτι ἄνθρωπος τὸ αὐτὸ καὶ μουσικὸς ἄνθρωπος· ἢ γὰρ ὅτι τῷ ἀνθρώπῳ μιᾷ οὔσῃ οὐσίᾳ συμβέβηκε τὸ μουσικόν, ἢ ὅτι ἄμφω τῶν καθ᾽ ἕκαστόν τινι συμβέβηκεν, οἷον Κορίσκῳ. πλὴν οὐ τὸν αὐτὸν τρόπον ἄμφω ὑπάρχει, ἀλλὰ τὸ μὲν ἴσως ὡς γένος καὶ ἐν τῇ οὐσίᾳ τὸ δὲ ὡς ἕξις ἢ πάθος τῆς οὐσίας.

> if it be bent, but continuous, is said to be one, just as each of [our] parts, as the leg or the arm.[24]

A difficulty arises here, since Aristotle supplies examples of things essentially one which are not essential beings. A bundle is not an essential being, nor are pieces of wood, even if they happen to be glued together. These examples also seem to differ from those of the bent line and the parts of a living thing. Aristotle continues: "Of these [continuous] things, that is *more* one which is continuous by nature than by art."[25] What could he mean by this? Continuity does not seem to differ in degree.

Importantly, Aristotle defines the continuous otherwise here than he does elsewhere. Most properly, the continuous is "that whose parts have a common limit," as opposed to what is contiguous, or touching, whose limits are merely together.[26] Here, however, he says: "That is called continuous whose motion is one through itself and cannot be otherwise: and [motion] is one when it is indivisible, that is, indivisible according to time."[27]

Sticks tied or glued together in a bundle,[28] then, as well as a bent line or the parts of a living thing,[29] are continuous in being moved as a whole. Given this sense of *continuity*, Aristotle's grouping together things which are continuous by art and things which are continuous by nature makes more sense. Furthermore, Aristotle himself, as we just noted, makes an additional distinction between the unity of the sticks and the parts of a living body: for what is one in this way by nature is more one than what is so by art. Aquinas elaborates in his *Commentary*:

> In things which are naturally continuous, the unity, through which continuity comes about, is not extraneous to the nature of the thing which becomes continuous through it [the unity], as happens among those things which are one by means of art, in which the chain, or the glue, or some such thing, is altogether extraneous to the nature of the things which have been joined

[24] Ibid., V 6 1015b3–1016a2. τῶν δὲ καθ᾽ ἑαυτὰ ἓν λεγομένων τὰ μὲν λέγεται τῷ συνεχῆ εἶναι, οἷον φάκελος δεσμῷ καὶ ξύλα κόλλῃ· καὶ γραμμή, κἂν κεκαμμένη ᾖ, συνεχὴς δέ, μία λέγεται, ὥσπερ καὶ τῶν μερῶν ἕκαστον, οἷον σκέλος καὶ βραχίων.

[25] Ibid., V.6.1016a3–4. αὐτῶν δὲ τούτων μᾶλλον ἓν τὰ φύσει συνεχῆ ἢ τέχνῃ.

[26] See *Categoriae*, 6.

[27] *Metaphysica*, V 6 1016a5–6. συνεχὲς δὲ λέγεται οὗ κίνησις μία καθ᾽ αὑτὸ καὶ μὴ οἷόν τε ἄλλως· μία δ᾽ οὗ ἀδιαίρετος, ἀδιαίρετος δὲ κατὰ χρόνον.

[28] These are continuous, and so *one*, by art.

[29] These are continuous, and so *one*, by nature.

> together. And thus those which have come together naturally are more like things which are essentially continuous, which are one to the highest degree.[30]

We said this above: certain things are one in virtue of themselves[31] (as the parts of a spider's body), while others are so only in virtue of something extrinsic to them (as a group of cows, whose only unity consists in their being *considered* as a group.)

In line with Plato (and in the spirit, more recently, of Frege), Aristotle distinguishes between what some would regard as a number—a collection or aggregate of units (a *heap*)—and number as a unified reality, as itself a *one*. Of course, there is a real difficulty in insisting that what is necessarily many (a number is, after all, a plurality) as one. Aristotle would be the first to insist that nothing can be actually one and actually many at the same time and in the same respect.

An aggregate, group, or heap, is not *really* one. Democritus was correct: it *is* impossible to get one from two. If number itself were a mere plurality of units, if lengths, surfaces, and mathematical solids were parts added to parts and nothing more—if the mathematicals had no *intrinsic* unity of their own, they would not be *per se* at all. An Aristotelian philosophy of mathematics makes a sharp distinction between quantities as collections made up of actually distinct parts and quantities which are their own realities. Partitive, yes, but also one in the strict sense. By Aristotle's account, the mathematicals must be, and each must be essentially one.

Where can the quantity we call *number* be found, then? Where *such* a one-in-the-many is found.[32] A bundle of sticks tied together *is* undeniably a single bundle, so much so that we also count bundles, which would be impossible had they not some kind of unity.[33] Yet this is not the *one*-in-the-many we are looking for. Its parts are simply touching and do not form an

[30] *In Metaphysicorum*, V 7 851. "In his quae sunt continua per naturam, illud unum, per quod fit continuatio, non est extraneum a natura rei quae per ipsum continuatur, sicut accidit in his quae sunt unum per artificium, in quibus vinculum, vel viscus, vel aliquid tale est omnino extraneum a natura colligatorum. Et ita ea quae sunt naturaliter colligata, prius accedunt ad ea quae sunt secundum se continua, quae sunt maxime unum."

[31] Let us recall that essentially one is our way of rendering the more literal one according to itself, in place of Aristotle's original καθ᾽ αὑτό (and Aquinas' *per se*).

[32] Not as celebrated as the one-in-the-many which leads us to the Forms, but analogous to it.

[33] This is Aristotle's point in saying that bundles have unity in the first place.

intrinsically continuous whole. In other words, *the parts being actually present*, we do not have something *one* in the strictest sense but a simple *many*. Number is not present here.[34] Granted, the sticks are less *discontinuous* than if they were scattered, but this is extrinsic. In contrast, things which are essentially one (like living substances) have parts which are unified intrinsically. Are two cows one body? Clearly not. They even lack unified motion. What, then, of a spider? How are the parts of a spider one?

If a spider is one substance, and if parts are actually distinct in virtue of substances being actually distinct, then the spider's parts are essentially one. *Such* unity is lacking in a bundle of sticks, and, *a fortiori*, in a group of cows (or cows' legs) which are not even in contact—after all, they are not naturally a group at all, *we* group them.

Beyond addressing what is one *per se,* mathematics must deal with and consider unity of another sort, namely, the *mathematical* unit.[35] If we are to establish the properties of numbers in arithmetic, for example, we need to know both that the numbers in question are one, as well as how its units are a distinct kind of *one* in their own right.

The meaning of *one* suited to our purpose here is *the indivisible*, the most proper sense of the term.

> In general those things which cannot be divided, insofar as they cannot, are said to be one, for example, if something cannot be divided as man, it is one man, and if [it cannot be divided] as animal, it is one animal, and if [it cannot be divided] as a magnitude, it is one magnitude … In every instance the one is indivisible either in quantity or in kind.[36]

[34] For more on Aristotle's insistence that a number be essentially one, see *Metaphysica*, VIII 3 and XIII 8.

[35] How and to what extent mathematicians do this will become clear soon.

[36] *Metaphysica,* V 6 1016b4–6, 25. καθόλου γὰρ ὅσα μὴ ἔχει διαίρεσιν, ᾗ μὴ ἔχει, ταύτῃ ἓν λέγεται, οἷον εἰ ᾗ ἄνθρωπος μὴ ἔχει διαίρεσιν, εἷς ἄνθρωπος, εἰ δ᾽ ᾗ ζῷον, ἓν ζῷον, εἰ δὲ ᾗ μέγεθος, ἓν μέγεθος. … πανταχοῦ δὲ τὸ ἓν ἢ τῷ ποσῷ ἢ τῷ εἴδει ἀδιαίρετον. Cf. *In Metaphysicorum*, V 8866, 874. "Things which are altogether indivisible are said to be one in the highest degree: for all the other ways [in which something is said to be one] are reduced to this way, since it is universally true that whatever things do not have division are said to be one insofar as they do not have division … Among all of these [different types of unit] there is this in common: that the first measure is indivisible according to quantity or in species." "Illa quae sunt penitus indivisibilia, maxime dicuntur unum: quia ad hunc modum omnes alii modi reducuntur, quia universaliter hoc est verum, quod quaecumque non habent divisionem, secundum hoc dicuntur unum, inquantum divisionem non habent. … In omnibus tamen istis hoc est commune, quod illud, quod est prima mensura, est indivisibile secundum quantitatem, vel secundum speciem."

As to what is properly *quantitative* there are two such: the principle of continuous quantity and that of discrete quantity. "That which is in no way divisible in quantity is the point or the unit: if without position, the unit, if having position, the point."[37]

Focusing on the mathematical unit, Aristotle notes that there is more to it than indivisibility: for it is naturally also a measure. Thus Aristotle:

> Now the one is opposed to the many, among numbers, as a measure [is opposed] to what is measured: but these [are opposed] as correlatives which are not such through themselves … Plurality is in a way the genus of number, since number is a plurality measured by the unit.[38]

Aquinas elaborates upon Aristotle's observation, setting our consideration of *this* sense of *the one* alongside its other senses:

> Plurality or multitude taken absolutely, which is opposed to the one which is converted with being, is, as it were, the genus of number; for a number is nothing else than a plurality or multitude measurable by the unit. Therefore the unit, according as it simply is called an indivisible being, is converted with being. But according as it takes on the notion of a measure it is determined to a certain genus of quantity, in which the notion of measure is properly found.[39]

[37] *Metaphysica*., V 6 1016b29–31. τὸ δὲ μηδαμῇ διαιρετὸν κατὰ [30] τὸ ποσὸν στιγμὴ καὶ μονάς, ἡ μὲν ἄθετος μονὰς ἡ δὲ θετὸς στιγμή.

[38] Ibid., X 6 1056b33–34, 1057a3. ἀντίκειται δὴ τὸ ἓν καὶ τὰ πολλὰ τὰ ἐν ἀριθμοῖς ὡς μέτρον μετρητῷ: ταῦτα δὲ ὡς τὰ πρός τι, ὅσα μὴ καθ᾽ αὑτὰ τῶν πρός τι. … τὸ δὲ πλῆθος οἷον γένος ἐστὶ τοῦ ἀριθμοῦ: ἔστι γὰρ ἀριθμὸς πλῆθος ἑνὶ μετρητόν.

[39] *In Metaphysicorum* X, 8, 2090. "Pluralitas sive multitudo absoluta, quae opponitur uni quod convertitur cum ente, est quasi genus numeri; quia numerus nihil aliud est quam pluralitas et multitudo mensurabilis uno. Sic igitur unum, secundum quod simpliciter dicitur ens indivisibile, convertitur cum ente. Secundum autem quod accipitur rationem mensurae, sic determinatur ad aliquod genus quantitatis, in quo proprie invenitur ratio mensurae." Aristotle's text: "Nothing prevents the unit from being less than something, for example, two: for if it is less it need not also be a few. Plurality is in a way the genus of number, since number is a plurality measured by the unit. And the unit and number are opposed somehow, not as contraries, but as certain relative terms were said to be: for they are opposed as a measure and what is measured. Whence not everything which may be one is a number, such as something which is indivisible." *Metaphysica*, X, 6, 1057a2–8. τὸ δὲ ἓν ἔλαττον εἶναι τινός, οἷον τοῖν δυοῖν, οὐδὲν κωλύει: οὐ γάρ, εἰ ἔλαττον, καὶ ὀλίγον. τὸ δὲ πλῆθος οἷον γένος ἐστὶ τοῦ ἀριθμοῦ: ἔστι γὰρ ἀριθμὸς πλῆθος ἑνὶ μετρητόν, καὶ ἀντίκειταί πως τὸ ἓν καὶ ἀριθμός, οὐχ ὡς ἐναντίον ἀλλ᾽ ὥσπερ εἴρηται τῶν πρός τι ἔνια: ᾗ γὰρ μέτρον τὸ δὲ μετρητόν, ταύτῃ ἀντίκειται, διὸ οὐ πᾶν ὃ ἂν ᾖ ἓν ἀριθμός ἐστιν, οἷον εἴ τι ἀδιαίρετόν ἐστιν.

Aristotelian quantities are similar both to the (mathematical) continuum and to physical compounds. The continuum must have parts for certain results to obtain, even if these parts are not actually present.[40] A physical compound must contain its elements in some way yet not *actually* (otherwise there would be no difference between the properties we observe in mixtures and those we observe in compounds having the same elements). So, too, for mathematicals. For example, a number does not actually contain the units of which it is composed, otherwise, as we have said, it would be a heap, arbitrary, and one merely extrinsically. This is the metaphysical insight into unity and plurality that undergirds Aristotle's distinctive account of the mathematicals:

> Substance cannot be made up of substances as actually present: for what is actually two can never be actually one, though if it is potentially two it will be [actually] one, for example, the double is made up of two halves only potentially: for the actuality would separate [the halves]. So that if substance is one, it will not be constituted of substances in this way [as actually present in it], as Democritus rightly says. For it is impossible (he says) for one to come from two or for two to come from one: for he makes his atoms substances. This is likewise clear with number, if, in fact, a number is a composition of units, as is said by some: for either two is not one, or one is not actually present in it [two].[41]

Spiders' parts are distinct-though-unified naturally, not arbitrarily, further support of which is that their bodies are *organic*—each part (or kind of part) serves as an instrument for a distinct purpose.[42] It is not surprising that living things provide us with the clearest instances of substances having number as an accident—perhaps the first, best examples we have of numbers are found in the digits on our hands.

[40] Among the consequences of assuming that a continuum is composed of actual parts are the many paradoxes associated with actually infinite quantities.

[41] *Metaphysica*, VII 13 1039a3–14. ἀδύνατον γὰρ οὐσίαν ἐξ οὐσιῶν εἶναι ἐνυπαρχουσῶν ὡς ἐντελεχείᾳ· τὰ γὰρ δύο οὕτως ἐντελεχείᾳ οὐδέποτε ἓν ἐντελεχείᾳ, ἀλλ᾽ ἐὰν δυνάμει δύο ᾖ, ἔσται ἕν (οἷον ἡ διπλασία ἐκ δύο ἡμίσεων δυνάμει γε: ἡ γὰρ ἐντελέχεια χωρίζει), ὥστ᾽ εἰ ἡ οὐσία ἕν, οὐκ ἔσται ἐξ οὐσιῶν ἐνυπαρχουσῶν καὶ κατὰ τοῦτον τὸν τρόπον, ὃν λέγει Δημόκριτος ὀρθῶς: ἀδύνατον γὰρ εἶναί φησιν ἐκ [10] δύο ἓν ἢ ἐξ ἑνὸς δύο γενέσθαι: τὰ γὰρ μεγέθη τὰ ἄτομα τὰς οὐσίας ποιεῖ. ὁμοίως τοίνυν δῆλον ὅτι καὶ ἐπὶ ἀριθμοῦ ἔξει, εἴπερ ἐστὶν ὁ ἀριθμὸς σύνθεσις μονάδων, ὥσπερ λέγεταί ὑπό τινων: ἢ γὰρ οὐχ ἓν ἡ δυὰς ἢ οὐκ ἔστι μονὰς ἐν αὐτῇ ἐντελεχείᾳ. See also *Ibid*., VIII 3.1044a3–5 and In Metaphysicorum, VIII 3 1725.

[42] *Organon* (ὄργανον) equates to *instrument* or *tool*.

Finally, then, is the spider eight in number, or are its legs? From what we have said, as the legs are not distinct substances, then number cannot separately exist in them: hence it must exist in the spider. But clearly, it does not exist in the spider *according to all that it is*, otherwise we would be compelled to say that a spider, having eight legs, is eight spiders. We might be pale, but not according to all that we are (else human nature itself would be pale). We are pale only in virtue of the surface of our skin. In a similar way, eight is in the spider, not according to everything that it is, but still intrinsically—in virtue of the distinction to be found among its legs.

This analysis is Aristotelian: as we have seen, the mathematicals are not collections of parts, rather, each is essentially, not accidentally, one.[43] Numbers differ from one another in kind. However true it is to say that someone who has six has more than someone who has five, six is not *more five*: the numbers are specifically distinct.[44] An implication of this doctrine is that a number, though equal to, is not identical to, the sum of its units. Five *equals* but is not *identical to* 1 + 1 + 1 + 1 + 1. The latter is a collection of actual units, one only accidentally, while the former is a single thing, an essential unity, which contains its units only virtually.[45]

It remains to point out that some countable things (e.g., the cows in the field,) do not contain number, since they lack the unity requisite for number itself. Others, having parts which are countable, also appear to have number as an accident, for they are parts of a substantial whole, and the parts of a substance are not actual substances themselves. The *countability* of arbitrary groupings (or the *measurability* of merely contiguous lengths, surfaces, and solids) has definite implications for later chapters, especially as regards mathematical realism as up against anti-realist accounts.

[43] Hence the seemingly paradoxical statement that two is one.

[44] As Aristotle says: "Just as a number from which a component is subtracted or added is no longer the same number, but different, even if what is subtracted or added be very small, so a definition or essence from which something is subtracted or added [is no longer the same]." Metaphysica, VIII 3 1043b36–1044a3. καὶ ὥσπερ οὐδ᾽ ἀπ᾽ ἀριθμοῦ ἀφαιρεθέντος τινὸς ἢ προστεθέντος ἐξ ὧν ὁ ἀριθμός ἐστιν, οὐκέτι ὁ αὐτὸς ἀριθμός ἐστιν ἀλλ᾽ ἕτερος, κἂν τοὐλάχιστον ἀφαιρεθῇ ἢ προστεθῇ, οὕτως οὐδὲ ὁ ὁρισμὸς οὐδὲ τὸ τί ἦν εἶναι οὐκέτι ἔσται ἀφαιρεθέντος τινὸς ἢ προστεθέντος.

[45] We will directly address the importance of virtual presence for Aquinas' philosophy of mathematics in Chap. 7.

This, then, is Aristotle's account of the mathematicals, that is, with his own answer to the questions whether the mathematicals really exist and where they can be found. Three points remain to be briefly introduced here. Their final resolution carries with it what remains in our account of Aristotle's mathematical philosophy.

Idealizing the Mathematicals

The mathematicals are *accidents* belonging to physical substances. Barring platonism, this seems the only way to support Aristotle's contention (which we will directly address in the next chapter) that systematic thought about the mathematicals takes the form of sciences—*episteme*—in a very strict sense. *Episteme* is only about real things. But is it *those* quantities the mathematician studies? Earlier arguments suggest so.[46] On the other hand, some mathematical objects as we find them physically do lack the character we ascribe to them in mathematics—Aristotle himself observes this:

> …sensible lines are not of the sort of which the geometer speaks (for no sensible thing is straight or curved in that way: for a [sensible] circle does not touch a bar at a point, but as Protagoras said [it did] in refuting the geometers).[47]

Likewise, later in the *Metaphysics*:

> …about what sort of thing ought we to reckon the mathematician is concerned? Certainly not things here [in this world], for none of these is the kind of thing the mathematical sciences study.[48]

This observation, in this context, takes the form of the so-called *precision problem*. Why the need to establish the reality of numbers, lengths, surfaces, and mathematical solids, as accidents of physical substances—with all the associated problems and distinctions—if those attributes are

[46] Recall the problem he raises as to whether mathematics and natural philosophy even differ, given that they are about the *same things*.

[47] *Metaphysica*, III 2 998a1–4. οὔτε γὰρ αἱ αἰσθηταὶ γραμμαὶ τοιαῦταί εἰσιν οἵας λέγει ὁ γεωμέτρης (οὐθὲν γὰρ εὐθὺ τῶν αἰσθητῶν οὕτως οὐδὲ στρογγύλον: ἅπτεται γὰρ τοῦ κανόνος οὐ κατὰ στιγμὴν ὁ κύκλος ἀλλ᾽ ὥσπερ Πρωταγόρας ἔλεγεν ἐλέγχων τοὺς γεωμέτρας).

[48] *Metaphysica*, XI 1 1059b11–12. περὶ ποῖα θετέον πραγματεύεσθαι τὸν μαθηματικόν; οὐ γὰρ δὴ περὶ τὰ δεῦρο: τούτων γὰρ οὐθέν ἐστιν οἷον αἱ μαθηματικαὶ ζητοῦσι τῶν ἐπιστημῶ.

not really what mathematics is about? We even observed (at the end of the last section) that counting does not presuppose that some substance really has that number, existing and having unity *per se*. The arbitrariness of such acts of counting, that is, their wholly contingent basis, does seem ill-suited to the sort of rigor and certainty we claim for the mathematical sciences. On the other hand, perhaps that is the very point: there is something artificial, too human, in such acts of counting and subsequent systematic thought about the numbers we have actually produced. The precision-problem is a major point in favor of mathematical anti-realism.

How Do the Mathematicals Exist *Materially*?

The last two books of Aristotle's *Metaphysics* supply his answer to the fifth *aporia* from Book Three: "Is there one kind of substance or more than one, as those who speak of the Forms and the intermediates, which the mathematical [sciences] are about, claim?"[49] While Aristotle devotes the major portion of these final books to a consideration, and refutation, of Plato's mathematical views, he does supply an indication of his own resolution early on:

> Man, as man, is one and indivisible: and [the arithmetician] supposes he is one and indivisible, and then investigates whether something is attributed to man as indivisible, while the geometrician [investigates man] neither as man nor as indivisible but as a solid. For, clearly, whatever would have belonged to man even if he somehow were not indivisible can belong to him apart from these things. For this reason geometricians speak rightly and treat of things which exist. And they *do exist*: for beings exist in two ways, namely, actually, and materially.[50]

Aristotle believes he has refuted one of Plato's claims, that the mathematicals are substances, existing separately from sensible ones. The

[49] Ibid., III 2997a35–b2. καὶ πότερον μοναχῶς ἢ πλείω γένη τετύχηκεν ὄντα τῶν οὐσιῶν, οἷον οἱ λέγοντες τά τε εἴδη καὶ τὰ μεταξύ, περὶ ἃ τὰς μαθηματικὰς εἶναί φασιν ἐπιστήμας;

[50] *Metaphysica*, XIII 3 1078a23–31. Emphasis mine. ἓν μὲν γὰρ καὶ ἀδιαίρετον ὁ ἄνθρωπος ᾗ ἄνθρωπος· ὁ δ᾽ ἔθετο ἓν ἀδιαίρετον, εἶτ᾽ ἐθεώρησεν εἴ τι τῷ ἀνθρώπῳ συμβέβηκεν ᾗ ἀδιαίρετος. ὁ δὲ γεωμέτρης οὔθ᾽ ᾗ ἄνθρωπος οὔθ᾽ ᾗ ἀδιαίρετος ἀλλ᾽ ᾗ στερεόν. ἃ γὰρ κἂν εἰ μή που ἦν ἀδιαίρετος ὑπῆρχεν αὐτῷ, δῆλον ὅτι καὶ ἄνευ τούτων ἐνδέχεται αὐτῷ ὑπάρχειν [τὸ δυνατόν], ὥστε διὰ τοῦτο ὀρθῶς οἱ γεωμέτραι λέγουσι, καὶ περὶ ὄντων διαλέγονται, καὶ ὄντα ἐστίν· διττὸν γὰρ τὸ ὄν, τὸ μὲν ἐντελεχείᾳ τὸ δ᾽ ὑλικῶς. This Chapter in Book Thirteen reveals much of what is to follow in our account of Aristotle.

implication, however, (assuming the mathematicals are real) is not without its own problems. One is the precision problem. Here, another surfaces: the distinction between actual and *material* is meant to resolve a difficulty, but it is not immediately apparent, in the context of *Metaphysics* XIII 3, what that problem is. What could it mean to say that the mathematicals exist, not actually but materially, in sensible things? Saying this is certainly *compatible* with what we have said so far, that the mathematicals, for Aristotle, are accidents inhering in sensible substances. Moreover, Aristotle does have a coherent account of *being* in multiple senses, one of which, being *potential*, he does contrast with being *actually*.[51] The need for the distinction, at this point, however, is unclear.

Realist or Non-Realist?

That all scientific knowledge proceeds from preexistent knowledge is the first claim Aristotle makes in his *ex professo* treatment of *episteme* (of which the mathematical sciences are one sort).[52] Before we can conclude, we must already know that our premises are true, whether our subject exists, and the subject and predicate terms are.

Here is the problem. Apart from whether straight lengths and plane surfaces—whether absolute quantitative regularity of *any* sort—obtain among physical substances (the precision problem), is it *necessary* that there (1) be some substance which actually contains twenty-eight parts, for example, that we (2) know of this substance, and that we (3) consider the number twenty-eight while ignoring its sensible aspects in that substance, before we can prove that *twenty-eight is a perfect number*?

This example may even *understate* the difficulty. Consider René Descartes' famous discussion of the chiliagon—a polygon with one-thousand sides. It is easy to prove that a regular chiliagon has bilateral symmetry, but we certainly do not *wait for* that conclusion until we have discovered a substance having that particular shape.

Construction is necessary to the mathematical method. Aristotle observes this in several places, for example, in his treatment of the infinite in the *Physics*, he says: "It is always possible to think of a greater [number]:

[51] See, for example, *Metaphysica*, V 7 for the distinction and especially Ibid., IX entire for its elucidation.

[52] Cf. Aristotle, *Analytica Posteriora*, (Oxford: Oxford University Press, 1982), I 1, hereafter *Analytica Posteriora*.

for the bisections of a magnitude are infinite."[53] Both his proposed solution to the *Meno dilemma* and his treatment of potentiality in geometry focus upon the necessity of mathematical constructions.[54]

Much as we observed that even substances *not containing a number* can be counted (along with the troubling question as to why an account of the mathematicals as accidents of physical substances is needed), so, too, here. If constructability is necessary to mathematical discoveries—at least to some, that is, to those attributes we demonstrate of quantities we have never seen—why say mathematics is not ultimately fictional? That same aspect of *we make* to which Plato and the other platonists from Chap. 1 objected seems foursquare *within* the mathematical realm. As we said above, what need for us to verify the reality of a substance having twenty-eight natural parts *before* we conclude that twenty-eight is a perfect number? Such a thing seems entirely unnecessary.

Beyond the nature of the mathematicals themselves (a metaphysical question, as it were) we now turn to the second major requirement for Aristotle's—for anyone's—coherent account of mathematical thought: mathematical epistemology. How do we have intellectual access to such realities and what form does such thinking take?

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[53] *Physica*, III 7 207b10–12. ἐπὶ δὲ τὸ πλεῖον ἀεὶ ἔστι νοῆσαι· ἄπειροι γὰρ αἱ διχοτομίαι τοῦ μεγέθους.

[54] Cf. *Analytica Posteriora*, I 1 and *Metaphysica*, IX 9, respectively.

CHAPTER 4

Aristotle on the Speculative and Middle Sciences

The Practical and Speculative Sciences

Aristotle famously presents us with one of the earliest, systematic accounts of what, and to what degree, we are able to know.[1] As he sees it, this natural drive gives rise to observations, claims, and complete systems of thought, each having varying degrees of support or evidence and certitude in the result. While he makes mention of this fundamental capacity throughout his writings, it is in his logical works that we see his account worked out most systematically. We begin with his observation that, among the things we can know, one such body of truths, mathematics, is distinguished from some others in being called a *science*.[2] First, then, we turn to Aristotle's account of *episteme*, that is, *science* or *scientific knowledge*.

[1] Not surprisingly, the man whom Dante calls *the master among those who know* and Aquinas *the Philosopher* famously begins his *Metaphysics* with the observation that all men naturally desire to know.

[2] αἵ μαθηματικαὶ τῶν ἐπιστημῶν Explicit references are common. Cf. *Analytica Posteriora* I 1, 14 and *Metaphysica,* III 2, IV 1, VI 1, XI 1 and 7, XII 8, and XIII 2–3.

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Aristotle's most direct treatment of *episteme* is found in *Posterior Analytics*.[3] After offering his solution to the so-called *Meno dilemma*[4] in the very first chapter, Aristotle turns to enumerating the conditions under which we can be said to know in this special way:

> We think we know [ἐπίστασθαι] some thing simply speaking (not accidentally, as a sophist does,) when we think we know the cause of that thing, that it is the cause, and that it cannot be otherwise.[5]

As for the construct whereby we know, Aristotle terms it a *demonstration*:

> I call such a 'scientific' syllogism [συλλογισμὸν ἐπιστημονικόν] a 'demonstration,' and by 'scientific' I mean [a syllogism] whose very possession makes us know.[6]

To this point, Aristotle envisions a type of syllogistic reasoning, a demonstration, which yields necessary knowledge of some thing through its causes.[7] Still, as Aristotle himself notes, "all rational teaching and learning come

[3] The term itself, or one of its forms, appears dozens of times in the course of the treatise.

[4] Meno's dilemma is famously raised in the dialogue of the same name.

> Meno: How will you inquire, Socrates, into that about which you do not know at all what it is? What sort of thing will you be looking for among the things you do not know? And even if you were to find it, how can you know that this is the thing which you did not know?
>
> Socrates: I know what you are trying to say, Meno. You do realize what a contentious argument you are introducing, that a man can inquire neither into what he knows nor into what he does not know: for he inquires either into what he knows, and so has no need to inquire, or into what he does not know, and is looking for he knows not what." Plato, "Meno", trans. W.K.C. Guthrie in *The Collected Dialogues of Plato*, eds. Edith Hamilton and Huntington Cairns (New Jersey: Princeton University Press, 1982), 80d–e, 363. Socrates, of course, responds with his famous account of recollection. Aristotle's own solution differs, as we will see, which difference has implications for his philosophy of mathematics.

[5] *Analytica Posteriora*, I 2 71b9–12. Ἐπίστασθαι δὲ οἰόμεθ᾽ ἕκαστον ἁπλῶς, ἀλλὰ μὴ τὸν σοφιστικὸν τρόπον τὸν κατὰ συμβεβηκός, ὅταν τήν τ᾽ αἰτίαν οἰώμεθα γινώσκειν δι᾽ ἣν τὸ πρᾶγμά ἐστιν, ὅτι ἐκείνου αἰτία ἐστί, καὶ μὴ ἐνδέχεσθαι τοῦτ᾽ ἄλλως ἔχειν.

[6] Ibid., I 2 71b17–19. ἀπόδειξιν δὲ λέγω συλλογισμὸν ἐπιστημονικόν· ἐπιστημονικὸν δὲ λέγω καθ᾽ ὃν τῶι ἔχειν αὐτὸν ἐπιστάμεθα.

[7] To know in this way is to provide an answer to the question *why*, the fourth (and last) among the questions Aristotle enumerates later in the *Posterior Analytics*, and to which he refers in other works. See, for example, *Ibid.,* II 1–2 and *Physica,* II 3 for Aristotle on the character and importance of causal knowledge.

about through preexistent knowledge."[8] Indeed, Aristotle spends the whole of the first chapter in *Posterior Analytics* establishing this proposition and setting out the conditions for, and types of, preexistent knowledge needed to teach and learn. Since demonstration is also a type of teaching and learning, it follows that such necessary and causal knowledge itself presupposes that we know other things, and even that we know them otherwise than demonstratively.[9]

What must we already know in order to teach and learn? We must already know that some things are so, that is, are true, we must know what certain terms mean, and we must know that the subject under consideration exists.

> It is necessary to have preexistent knowledge of two sorts: for in some cases it is necessary to assume that something is, in others what the term is, and in still others both. For example, of *concerning everything, either the affirmation or the denial is true* [we assume] that it is, of *triangle* [we assume] what it means, and of *unit* [we assume] both what it means and that it is.[10]

Recalling that a demonstration is a kind of syllogism, then, Aristotle sets out the kinds of preexistent knowledge required for it in syllogistic terms:

> Now, if scientific knowledge [τὸ ἐπίστασθαι] is as we have said, it is necessary that [the premises] of demonstrated knowledge be true, first, and immediate, as well as better-known than, prior to, and causes of, the conclusion.[11]

Aristotle next justifies each of the conditions in light of his definition of demonstration. One very important point, for our purposes, is that the first

[8] *Analytica Posteriora*, I 1 71a1–2. Πᾶσα διδασκαλία καὶ πᾶσα μάθησις διανοητικὴ ἐκ προϋπαρχούσης γίνεται γνώσεως.

[9] It turns out that this *other way* is induction: "So it is clear that we must come to know the first [principles] by induction." *Analytica Posteriora*, II 19 100b3–4. δῆλον δὴ ὅτι ἡμῖν τὰ πρῶτα ἐπαγωγῇι γνωρίζειν ἀναγκαῖον.

[10] *Analytica Posteriora*, I 1 71a11–16. διχῶς δ᾽ ἀναγκαῖον προγινῶσκειν· τὰ μὲν γάρ, ὅτι ἐστι, προϋπολαμβάνειν ἀναγκαῖον, τὰ δέ, τί τὸ λεγόμενόν ἐστι, ξυνιέναι δεῖ, τὰ δ᾽ ἄμφω, οἷον ὅτι μὲν ἅπαν ἢ φῆσαι ἢ ἀποφῆσαι ἀληθές, ὅτι ἐστι, τὸ δὲ τρίγωνον, ὅτι τοδὶ σημαίνει, τὴν δὲ μονάδα ἄμφω, καὶ τί σημαίνει καὶ ὅτι ἔστιν. It is in the course of his argument for this contention that we find Aristotle's answer to the above-mentioned *Meno dilemma*.

[11] Ibid., I 2 71b19–22. εἰ τοίνυν ἐστὶ τὸ ἐπίστασθαι οἷον ἔθεμεν, ἀνάγκη καὶ τὴν ἀποδεικτικὴν ἐπιστήμην ἐξ ἀληθῶν τ᾽ εἶναι καὶ πρώτων καὶ ἀμέσων καὶ γνωριμωτέρων καὶ προτέρων καὶ αἰτίων τοῦ συμπεράσματος.

principles of demonstrations must be true.[12] He argues that what is not true cannot be known:

> [The premises] must be true, because it is not possible to know [ἐπίστασθαι] what does not exist, for example, [it is not possible to know] that the diagonal [of a square] is commensurate [with its side].[13]

While Aristotle is not insisting that knowledge is possible only when we speak of things that are real,[14] still, this type, *episteme*, does require that what is in our mind conform to the way things are.[15]

Not surprisingly, then, Aristotle is a realist when it comes to scientific knowledge, *episteme*—it is a kind of speculative knowledge, after all, so truth itself is a condition which it must meet to be knowledge of this sort.

In *Nicomachean Ethics* VI, Aristotle gives a precise account of virtues which do not make us good people, as such but only good in some way or other—the *intellectual* virtues:

> Virtue is also divided according to this difference: for we call some of the virtues 'intellectual' and others 'moral,' wisdom, judgment, and prudence being intellectual, generosity and temperance, moral. For when speaking of [someone's] character we do not say that he is wise or has good judgment but that he is mild or temperate. Still, we praise the wise man on account of [the knowledge] he has, and such praiseworthy things we call 'virtues'.[16]

[12] Indeed, much of the modern controversy surrounding the philosophy of mathematics comes down to this question.

[13] Ibid., I 2 71b25–26. ἀληθῆ μὲν οὖν δεῖ εἶναι, ὅτι οὐκ ἔστι τὸ μὴ ὂν ἐπίστασθαι, οἷον ὅτι ἡ διάμετρος σύμμετρος.

[14] Indeed, he says, of the conditions just laid out, that "there can be a syllogism without these, but it will not be a demonstration: for it will not produce scientific knowledge." Ibid., I 2 71b23–25. συλλογισμὸς μὲν γὰρ ἔσται καὶ ἄνευ τούτων, ἀπόδειξις δ᾽ οὐκ ἔσται· οὐ γὰρ ποιήσει ἐπιστήμην.

[15] Cf. Aristotle's well-known definitions of falsity and truth: "To say that what is, is not or that what is not, is, is false, while to say that what is, is and that what is not, is not, is true." *Metaphysica*, IV 7 1011b26–28. τὸ μὲν γὰρ λέγειν τὸ ὂν μὴ εἶναι ἢ τὸ μὴ ὂν εἶναι ψεῦδος, τὸ δὲ τὸ ὂν εἶναι καὶ τὸ μὴ ὂν μὴ εἶναι ἀληθές.

[16] Aristotle, *Ethica Nicomachea* (Oxford: Oxford University Press, 1979), I 13 1103a3–10, hereafter *Ethica* Nicomachea. διορίζεται δὲ καὶ ἡ ἀρετὴ κατὰ τὴν διαφορὰν ταύτην: λέγομεν γὰρ αὐτῶν τὰς μὲν διανοητικὰς τὰς δὲ ἠθικάς, σοφίαν μὲν καὶ σύνεσιν καὶ φρόνησιν διανοητικάς, ἐλευθεριότητα δὲ καὶ σωφροσύνην ἠθικάς. λέγοντες γὰρ περὶ τοῦ ἤθους οὐ λέγομεν ὅτι σοφὸς ἢ συνετὸς ἀλλ᾽ ὅτι πρᾶος ἢ σώφρων: ἐπαινοῦμεν δὲ καὶ τὸν σοφὸν κατὰ τὴν ἕξιν: τῶν ἕξεων δὲ τὰς ἐπαινετὰς ἀρετὰς λέγομεν.

As to *intellectual*, Aristotle elaborates:

> Let it be supposed that there are two parts having reason: one by which we contemplate things whose causes cannot be otherwise, the other by which we contemplate things [whose causes] can: for according to this difference in kind the parts of the soul naturally also differ in kind, since each has knowledge on account of a certain similarity and conformity. Let one of these be called 'scientific' and the other 'calculative': for to deliberate and to calculate are the same, but no one deliberates about what cannot be otherwise. Therefore the calculative is one of the parts [of the soul] having reason. Now we must decide what is the best habit of each: for this is its virtue.[17]

Aristotle eventually associates both the scientific part and the calculative part with truth,[18] though the truths themselves differ as being ordered, or not being ordered, to action. In this way, he comes to a division of the intellectual virtues in chapter three:

> There are five [habits] whereby the soul speaks the truth when affirming or denying, namely, art, scientific knowledge, prudence, wisdom, and understanding (for supposition and opinion can speak falsely.)[19]

Necessary and Sufficient Conditions for Episteme

Episteme[20] is distinguished from two other virtues among the kinds of speculative knowledge, whereas art and prudence are types of practical knowledge. In sum, through *episteme*, one possesses the truth concerning

[17] Ibid., VI 1 1139a6–16. καὶ ὑποκείσθω δύο τὰ λόγον ἔχοντα, ἓν μὲν ᾧ θεωροῦμεν τὰ τοιαῦτα τῶν ὄντων ὅσων αἱ ἀρχαὶ μὴ ἐνδέχονται ἄλλως ἔχειν, ἓν δὲ ᾧ τὰ ἐνδεχόμενα: πρὸς γὰρ τὰ τῷ γένει ἕτερα καὶ τῶν τῆς ψυχῆς μορίων ἕτερον τῷ γένει τὸ πρὸς ἑκάτερον πεφυκός, εἴπερ καθ᾽ ὁμοιότητά τινα καὶ οἰκειότητα ἡ γνῶσις ὑπάρχει αὐτοῖς. λεγέσθω δὲ τούτων τὸ μὲν ἐπιστημονικὸν τὸ δὲ λογιστικόν: τὸ γὰρ βουλεύεσθαι καὶ λογίζεσθαι ταὐτόν, οὐδεὶς δὲ βουλεύεται περὶ τῶν μὴ ἐνδεχομένων ἄλλως ἔχειν. ὥστε τὸ λογιστικόν ἐστιν ἕν τι μέρος τοῦ λόγον ἔχοντος. ληπτέον ἄρ᾽ ἑκατέρου τούτων τίς ἡ βελτίστη ἕξις: αὕτη γὰρ ἀρετὴ ἑκατέρου.

[18] This is the conclusion of Ibid., VI 2.

[19] Ibid., VI 3 1139b15–18. ἔστω δὴ οἷς ἀληθεύει ἡ ψυχὴ τῷ καταφάναι ἢ ἀποφάναι, πέντε τὸν ἀριθμόν: ταῦτα δ᾽ ἐστὶ τέχνη ἐπιστήμη φρόνησις σοφία νοῦς: ὑπολήψει γὰρ καὶ δόξῃ ἐνδέχεται διαψεύδεσθαι.

[20] We shall customarily use this term in place of *science* or *scientific knowledge* for the sake of clarity. As is commonly known, the modern sciences do not meet the conditions Aristotle sets out for *episteme*.

invariable matters by way of affirmation and denial. On the other hand, this is also true of understanding (*nous*) and philosophic wisdom (*sophia*), so there remains a further difference to discover, distinguishing *episteme* from all other forms of knowledge:

> Whence it is clear what episteme is (if, that is, one is being precise and not following similarities.) For we all suppose that what we know scientifically [ἐπιστάμεθα] cannot be otherwise: of things which can be otherwise, we do not know whether they [even] exist or not once they are out of our sight. Therefore what is scientifically known [τὸ ἐπιστητόν] is necessary. It is also, then, eternal: for all things that are necessary, simply speaking, are [also] eternal; eternal things are also ungenerated and incorruptible. Further, it seems that all episteme can be taught, and what we can know in that way can be learned. Now all teaching begins with what is already known (as we also say in the Analytics also): for it comes about either through induction or through a syllogism. Now induction is a principle of the universal itself, while a syllogism proceeds from the universal. Therefore, there are principles from which a syllogism proceeds for which there is no syllogism, and so [such principles are known] by induction. Episteme, then, is a demonstrative habit, having also the conditions we specify in the Analytics. For when one has episteme, he believes in a certain way and the principles are known to him: if they are not more [known] to him than the conclusion, he will have episteme only accidentally.[21]

To summarize. First, mathematics is identified as a kind of *episteme* by Aristotle.[22] *Episteme* itself is demonstratively knowing the truth concerning

[21] Ibid., VI 3 1139b18–35. ἐπιστήμη μὲν οὖν τί ἐστιν, ἐντεῦθεν φανερόν, εἰ δεῖ ἀκριβολογεῖσθαι καὶ μὴ ἀκολουθεῖν ταῖς ὁμοιότησιν. πάντες γὰρ ὑπολαμβάνομεν, ὃ ἐπιστάμεθα, μηδ᾽ ἐνδέχεσθαι ἄλλως ἔχειν: τὰ δ᾽ ἐνδεχόμενα ἄλλως, ὅταν ἔξω τοῦ θεωρεῖν γένηται, λανθάνει εἰ ἔστιν ἢ μή. ἐξ ἀνάγκης ἄρα ἐστὶ τὸ ἐπιστητόν. ἀίδιον ἄρα: τὰ γὰρ ἐξ ἀνάγκης ὄντα ἁπλῶς πάντα ἀίδια, τὰ δ᾽ ἀίδια ἀγένητα καὶ ἄφθαρτα. ἔτι διδακτὴ ἅπασα ἐπιστήμη δοκεῖ εἶναι, καὶ τὸ ἐπιστητὸν μαθητόν. ἐκ προγινωσκομένων δὲ πᾶσα διδασκαλία, ὥσπερ καὶ ἐν τοῖς ἀναλυτικοῖς λέγομεν: ἡ μὲν γὰρ δι᾽ ἐπαγωγῆς, ἡ δὲ συλλογισμῷ. ἡ μὲν δὴ ἐπαγωγὴ ἀρχή ἐστι καὶ τοῦ καθόλου, ὁ δὲ συλλογισμὸς ἐκ τῶν καθόλου. εἰσὶν ἄρα ἀρχαὶ ἐξ ὧν ὁ συλλογισμός, ὧν οὐκ ἔστι συλλογισμός: ἐπαγωγὴ ἄρα. ἡ μὲν ἄρα ἐπιστήμη ἐστὶν ἕξις ἀποδεικτική, καὶ ὅσα ἄλλα προσδιοριζόμεθα ἐν τοῖς ἀναλυτικοῖς: ὅταν γάρ πως πιστεύῃ καὶ γνώριμοι αὐτῷ ὦσιν αἱ ἀρχαί, ἐπίσταται: εἰ γὰρ μὴ μᾶλλον τοῦ συμπεράσματος, κατὰ συμβεβηκὸς ἕξει τὴν ἐπιστήμην.

[22] Of course, it is the implications of this claim for mathematics which is our primary aim in this chapter, as well as the work as a whole.

some matter or, when not actually considering it, the capacity to so demonstrate.[23] The matter under consideration is necessary, that is, it cannot be otherwise, nor can I be in error as long as I have *episteme*.

Pure Mathematics Is Episteme

We have already noted that Aristotle often explicitly speaks of mathematics as being among the sciences: αἵ μαθηματικαὶ τῶν ἐπιστημῶν.[24] This, then, is where Aristotle's philosophy of mathematics begins. Mathematics consists of demonstrations, syllogisms bringing us to necessary knowledge of mathematical conclusions, about which we already know a number of things even before we learn it—we know the truth of the principles upon which it is based—not in the sense that we have *assumed* their truth, but in the sense that we have *seen* their conformity with the ways things are. Again, one cannot know what does not exist. Such principles also lead us to see why mathematical conclusions are so, insofar as through them we come to see the causes of mathematical things, that they are their causes, and that they cannot be otherwise.

This account becomes even more clear when we consider the other areas of study Aristotle includes among the sciences—*the types of episteme*. We find his division in *Metaphysics* VI 1:

> Therefore there are three theoretical philosophies: mathematics, physics, and theology (since it is obvious that, if the divine exists anywhere, it exists in such a nature.)[25]

With physics likely being the most familiar example among these, we get a better sense of what Aristotle conceives mathematics to be. In short, its objects are real, *out there*, independent of our minds.[26]

[23] We nearly always think of a 'habit' as an internal tendency resulting from repeated like actions, but it has a much broader meaning for Aristotle, including what we mean by *possession*. Thus, before I know that 2 + 2 = 4, I cannot be said to know in any way other than potentially. At the moment I see its truth, I *have* knowledge (note the etymological connection to *habit* here,) nor do I cease to have it subsequently, as evidenced by my capacity to recall it as necessary. Of course, if I am unable to subsequently recall what I once knew but no longer know, I am again in the state of a learner.

[24] Romanized as hai mathematikai tōn epistēmōn.

[25] *Metaphysica*, VI 1 1026a17–20. ὥστε τρεῖς ἂν εἶεν φιλοσοφίαι θεωρητικαί, μαθηματική, φυσική, θεολογική (οὐ γὰρ ἄδηλον ὅτι εἴ που τὸ θεῖον ὑπάρχει, ἐν τῇ τοιαύτῃ φύσει ὑπάρχει).

[26] Recall the efforts of the mathematical realists (Chap. 1) to show precisely that.

Thinking about the mathematicals can fulfill the conditions Aristotle lays out for *episteme*—indeed, he explicitly includes αἵ μαθηματικαὶ τῶν ἐπιστημῶν among the ways we, in fact, do so.

Let us now recall the three difficulties we introduced at the end of the last chapter. We see and feel physical things in the course of our daily lives, but when has one ever encountered straight lines and circles?[27] Need the mathematicals which we are about actually exist, or does their mere possibility suffice? Does the need for constructive methods in mathematics render the claims and efforts of mathematical realists moot from the very start?

To continue in the same vein, one commonly proposed solution to the problems besetting mathematical realism is to insist that mathematical truth is achieved, not through the observation of independent realities, but through the successful application of these mental creations (numbers, lines, circles, and so on) to the real world.[28] In such an account, mathematical claims and conclusions are seen as successful fictions, their truth consisting in this very utility.[29]

Mathematics *is* useful. Indeed, between the two sorts of science Aristotle mentions, the scientific and the calculative, it seems to violate common sense to insist that mathematics, of all things, is not calculative. And yet, intent upon his claim that mathematics studies the real world, that the truth or falsity of mathematical claims lies in their conformity with some independent reality, Aristotle sees mathematics as scientific, as *episteme*.[30] How does the utility of mathematics not warrant its inclusion among the *calculative* virtues?

[27] The precision problem.

[28] There are many different sorts of mathematical anti-realists. Some believe the successful application of mathematics to physical problems implies nothing about the existence of mathematical objects, while others, though still unwilling to say that mathematical objects are real in the same sense as the physical objects they describe, believe it does.

[29] This is the view of Hartry Field, for example, whose *Science Without Numbers* is often taken as a point-of-origin for mathematical fictionalism. We will address fictionalism more directly in a later chapter.

[30] Bear in mind Aristotle's division of the intellectual virtues, first into scientific and calculative, then, respectively, into (properly) scientific knowledge, wisdom, and understanding, and then art and prudence. The question before us is: why would Aristotle not have placed the eminently useful mathematical sciences alongside art and prudence, themselves also eminently useful?

The Middle Sciences

One possibility is that mathematics, as *episteme*, can be *applied* to the physical world. Indeed, at least at first glance, Aristotle seems to be speaking of such an application (and so an associated utility) in the *Physics*. In the course of distinguishing physics from the coordinate science of mathematics, Aristotle speaks of a sort of *middle ground* between the two disciplines:

> The more physical among the mathematical sciences, for example optics, harmonics, and astronomy, also [make this difference] clear. They are, in a certain way, just the opposite of geometry. For geometry considers the physical line, but not as physical, while optics [considers] the mathematical line, not as mathematical, but as physical.[31]

Among the mathematical sciences, some are purely scientific in Aristotle's sense, in that their object is to learn what necessary conclusions can be known about their proper subject. Others, however, are *mixed in* with the physical sciences, in that their concern is not properly mathematical, but is rather an application of mathematical conclusions to the physical world. This becomes clear as we consider Aristotle's own examples.

Geometry is about coming to know the properties and attributes of continuous quantity (lines, surfaces, solids,) whereas optics, harmonics, and astronomy are inherently about light and sight, sounds and hearing, and heavenly bodies and their movements, respectively. Whereas the geometer would be interested in establishing the equality of angles about a certain line, say, one studying the properties of light would be more directly concerned with proving that the angle of light's incidence upon a surface is equal to its reflected angle. So, too, the arithmetician would be about establishing the truth of the composition of ratios (e.g., that the ratio of 2:3 when composed with the ratio of 3:4 yields the ratio of 1:2), whereas the musician would apply that truth to harmonies of sound, discovering that the musical interval called the fifth, whose essence consists of a ratio of 2:3, when compounded with the fourth, whose essence consists of a ratio

[31] *Physica*, II 2 194a7–12. δηλοῖ δὲ καὶ τὰ φυσικώτερα τῶν μαθημάτων, οἷον ὀπτικὴ καὶ ἁρμονικὴ καὶ ἀστρολογία· ἀνάπαλιν γὰρ τρόπον τιν' ἔχουσιν τῇ γεωμετρίᾳ. ἡ μὲν γὰρ γεωμετρία περὶ γραμμῆς φυσικῆς σκοπεῖ, ἀλλ' οὐχ ᾗ φυσική, ἡ δ' ὀπτικὴ μαθηματικὴν μὲν γραμμήν, ἀλλ' οὐχ ᾗ μαθηματικὴ ἀλλ' ᾗ φυσική.

of 3:4, yields an octave, whose essence consists of a ratio of 1:2.[32] The application of geometrical principles in astronomy is, likewise, an application of so-called *pure* mathematical conclusions to a particular, physical subject matter, in this case, the real or apparent movement of the planets and stars. Finally, an excellent and clear example can be supplied from experience, for anyone who has applied the Pythagorean theorem to assure that a building's foundation or walls have been set *square*, or at right angles to one another.

Aristotle does take stock of our thinking about the mathematicals *in a practical way*, something about which we all have an intuitive and clear sense. It is a valid question whether "the more physical among the mathematical sciences" are *even* scientific, in Aristotle's sense, that is, ordered to truth itself, and not to action or production.[33]

While we recognize the tendency to make the successful application of mathematical systems, the sole measure of their truth, in the manner we mentioned above, it is not the seemingly inherent utility of mathematics

[32] That is, one will discover in producing a musical fifth that it consists in a ratio of 2:3 in string lengths, volumes of vibrating air, or, even more fundamentally, vibrations per second. Similarly for the fourth and the numerical ratio of 3:4. The fifth added to the fourth produces an octave, an interval of sounds consisting in a ratio of 1:2 in string lengths, and so on. Aristotle gives the octave as his example of a formal cause when he treats of the four causes in *Ibid.*, II 3.

[33] Aquinas has an interesting observation under this heading: "The speculative and the practical intellect differ in this respect, that the speculative intellect looks at truth absolutely speaking, whereas the practical intellect looks at truth in an order to doing some work. Now it sometimes happens that the truth, considered in itself, cannot be regarded as a measure of doing some work, as happens in mathematics and in those which are separate from motion; so it happens that a consideration of this sort of truth is only in the speculative intellect. But other times a truth, considered in itself, can be regarded as a measure for doing some work: and in this case the speculative intellect is practical by way of extension to the doing of that work." Thomas Aquinas, *Scriptum Super Libros Sententiarum*, eds. P. Mandonnet and M. F. Moos, (Paris: Lethielleux, 1929, 1956), lib. 3 d. 23 q. 2 a. 3 qc. 2 co., hereafter *Scriptum Super Sententiarum*, "Ad secundam quaestionem dicendum, quod intellectus speculativus et practicus in hoc differunt quod intellectus speculativus considerat verum absolute, practicus autem considerat verum in ordine ad opus. Contingit autem quandoque quod verum ipsum quod in se considerabatur, non potest considerari ut regula operis, sicut accidit in mathematicis, et in his quae a motu separata sunt; unde hujusmodi veri consideratio est tantum in intellectu speculativo. Quandoque autem verum quod in re consideratur, potest ut regula operis considerari: et tunc intellectus speculativus fit practicus per extensionem ad opus."

that would sufficiently justify such a conclusion. Rather, one would naturally *expect* the mathematical sciences, as understood by Aristotle, to be successfully applied to that same physical reality which is the *source* of that truth. Indeed, the successful application of the mathematical sciences is among the best and most-enduring arguments *in favor of* mathematical realism.[34] In other words, Aristotle's insistence upon the strictly scientific, or pure, character of the mathematical sciences in no way *diminishes* their utility. It does raise the question, however, whether utility is a sufficient condition for the truth of a mathematical system of thought. Some mathematical philosophers point to utility as *the* marker of mathematical truth.[35] Here, though, Aristotle's account of "the more physical among the mathematical sciences" allows both mathematics of the pure sort and mathematics of the applied sort to be *true*, albeit in different ways. We return to Aristotle's distinction between the scientific and calculative sciences, this time focusing more closely upon how the truth of the one differs from the truth of the other with respect to *agency*.

The Pure Sciences Are Prior to the Applied

Book Ten of the *Nicomachean Ethics* returns to an issue Aristotle raises at the outset: which of our activities is the best, that is, in which does human happiness *consist*? While he makes a passing mention of the *contemplation* of truth at this early stage and delves into the speculative virtues of *episteme*, *nous,* and *sophia* in Book Five (as we noted above), he saves his direct treatment of contemplative happiness, θεωρία, for the work's end:

> If happiness is an activity in accordance with virtue, it is reasonable that it would [accord with] the highest [virtue]: which is of our best [part]. Whether

[34] Glimmers of this in more recent times can be seen in the so-called *indispensability argument* formulated by Putnam and Quine. See W.V.O. Quine, "Success and the Limits of Mathematization," in *Theories and Things* (Cambridge: Harvard University Press, 1981): 149–50 and "On What There Is," *Review of Metaphysics* 2, (Sep. 1948): 21–38 and Hilary Putnam, *Philosophy of Logic* (New York: Harper and Row, 1971): 57. We will address indispensability in Part III.

[35] Jonathan Lear, for example, takes the discovery of *bridges* between pure mathematics and the physical world as the sole warrant for our saying that any particular mathematical statement is true. See Jonathan Lear, "Aristotle's Philosophy of Mathematics," 190–91. We will address Lear's account in a later chapter.

> this [part] is the understanding, or what naturally rules and directs [us] and understands noble and divine things, whether it is itself divine or the most divine part of us, the activity of this part, in accordance with its proper virtue, will be complete happiness. We have already said that this [activity] is contemplative.[36]

Aristotle's way of distinguishing among human goods is marvelously insightful. Whereas *good*, *better*, and *best* appear to be ranked quantitatively—differing as to the *amount* of goodness, respectively—Aristotle sets the *more* and *less* aside in preference to another relation, namely, *cause* and *effect*, or, in this context, *means* and *ends*. Ranking our good actions and their own ends he says: "Where there is an end beyond certain activities, the result is *better than* the activities."[37] This is true of intellectual activities as of any other—indeed, Aristotle *begins* the work with the claim that "Every art and inquiry, likewise every action and choice, seems to aim at some good."[38]

Is there such an ordering between the *scientific* and *calculative* sciences as well? Between mathematics as pure and mathematics as applied? We know from the above that contemplation is the *best* human activity. Presumably that is because it is an end and not a means—we speak in such cases of *knowledge for its own sake*. In contrast with this, the very notion of a practical science sets it in order to some end. *Effective* or *successful* as sole criteria for mathematical *truth* seem not to take into account the ordering of such mathematical activities to mathematical truths of a higher order.[39]

So much, then, for Aristotle's account of the possibility and nature of our thinking about the mathematicals. In the next chapter we will turn to two aspects of Aristotle's mathematical philosophy that require a much closer look: (1) exactly how we acquire intellectual access to the

[36] *Ethica Nicomachea*, X 71177a10–19. εἰ δ' ἐστὶν ἡ εὐδαιμονία κατ' ἀρετὴν ἐνέργεια, εὔλογον κατὰ τὴν κρατίστην· αὕτη δ' ἂν εἴη τοῦ ἀρίστου. Εἴτε δὴ νοῦς τοῦτο εἴτε ἄλλο τι, ὃ δὴ κατὰ φύσιν δοκεῖ ἄρχειν καὶ ἡγεῖσθαι καὶ ἔννοιαν ἔχειν περὶ καλῶν καὶ θείων, εἴτε θεῖον ὂν καὶ αὐτὸ εἴτε τῶν ἐν ἡμῖν τὸ θειότατον, ἡ τούτου ἐνέργεια κατὰ τὴν οἰκείαν ἀρετὴν εἴη ἂν ἡ τελεία εὐδαιμονία. Ὅτι δ' ἐστὶ θεωρητική, εἴρηται.

[37] Ibid., I 11094a5–6. Emphasis mine. ὧν δ' εἰσὶ τέλη τινὰ παρὰ τὰς πράξεις, ἐν τούτοις βελτίω πέφυκε τῶν ἐνεργειῶν τὰ ἔργα.

[38] Ibid., 1094a1–2. πᾶσα τέχνη καὶ πᾶσα μέθοδος, ὁμοίως δὲ πρᾶξίς τε καὶ προαίρεσις, ἀγαθοῦ τινὸς ἐφίεσθαι δοκεῖ·

[39] Aristotle argues for the superiority of the speculative sciences over the practical sciences in many places. Cf. *Ethica Nicomachea* X 7, for instance.

mathematicals—(Aristotle calls this *separation*) and (2) his aforementioned reference to *matter* in mathematics. This will allow us to address the three difficulties we have kept in mind since the end of the last chapter and complete our account of Aristotle's mathematical philosophy.[40]

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[40] Those difficulties were: (1) We see and feel physical things in the course of our daily lives, but when has one ever encountered straight lines and circles? (2) Need the mathematicals which we are about actually exist, or does their mere possibility suffice? (3) Does the need for constructive methods in mathematics render the claims and efforts of mathematical realists moot from the very start?

CHAPTER 5

Aristotle on Abstraction and Intelligible Matter

Formal Abstraction: A Middle Way Between Platonism and Nominalism

We first raised the notion of *separation* in Aristotle's consideration of the difference between the mathematical sciences and the natural sciences from *Physics* II: "We must next ask how the mathematician differs from the philosopher of nature, for physical bodies have surfaces and volumes and lines and points, about which the mathematician is concerned."[1] Aristotle continues:

> Now the mathematician is also about such things [physical surfaces, volumes, and so on] but not insofar as they are the limits of physical bodies, nor does he think about such accidents as accidents: which is why he separates them. For they are separate from motion in thought, and it makes no difference, nor does any falsehood result, if they are separated. … This becomes clear when we try to state their definitions and the definitions of their accidents. For *odd* and *even*, *straight* and *curved*, as well as *number*, *line*, and

[1] *Physica*, II 2 193b23–24. μετὰ τοῦτο θεωρητέον τίνι διαφέρει ὁ μαθηματικὸς τοῦ φυσικοῦ (καὶ γὰρ ἐπίπεδα καὶ στερεὰ ἔχει τὰ φυσικὰ σώματα καὶ μήκη καὶ στιγμάς, περὶ ὧν σκοπεῖ ὁ μαθηματικός).

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> *figure*, are [defined] apart from motion, whereas *flesh* and *bone* and *man* are not—rather, we say these are [defined] like *snub nose* and not like *curved*.[2]

Note the difference between the sort of separation to which Aristotle refers here and the kind he ascribes to *those who say there are Forms* in the same passages. By Aristotle's account the Platonists attempt the impossible: separating inseparables. Excluding matter and motion, their very *physicality*, from our understanding of flesh, bone, and man, for example, results only in falsehood. On the other hand, to recognize that the definition of one thing does not necessarily include that of another—even if the two happen to exist together—is not *false*. The paleness of a healthy animal is not essentially different from another paleness in being *pale*. As we saw earlier, beings are not only *per se*, but also *per accidens*, and nothing prevents us from recognizing this intellectually, even if we might not succeed in *really* separating the one from the other. In short, fact is not necessity.

Aristotle supplies an extended treatment of mathematical separation (often called *abstraction*, which serves to distinguish it from a claim-separation error of the Platonist sort) near the beginning his *ex professo* treatment of the philosophy of mathematics.[3] We mentioned Chapter Three of Book Thirteen when we first listed our remaining difficulties, just mentioned. This chapter warrants a careful analysis, as it contains Aristotle's clearest explanation of mathematical abstraction. Remember, too, that it is by means of this close analysis, and our subsequent consideration of matter in mathematics, that we will resolve the remaining Aristotelian difficulties.

Aristotle begins by pointing out what he has just established regarding the Platonic mathematicals:

> We have made it sufficiently clear that [mathematicals] are neither substances to a greater extent than bodies nor really prior to sensible sub-

[2] Ibid., II 21093b30–35. περὶ τούτων μὲν οὖν πραγματεύεται καὶ ὁ μαθηματικός, ἀλλ'' οὐχ ᾗ φυσικοῦ σώματος πέρας ἕκαστον· οὐδὲ τὰ συμβεβηκότα θεωρεῖ ᾗ τοιούτοις οὖσι συμβέβηκεν· διὸ καὶ χωρίζει· χωριστὰ γὰρ τῇ νοήσει κινήσεώς ἐστι, καὶ οὐδὲν διαφέρει, οὐδὲ γίγνεται ψεῦδος χωριζόντων. … γίγνοιτο δ'' ἂν τοῦτο δῆλον, εἴ τις ἑκατέρων πειρῷτο λέγειν τοὺς ὅρους, καὶ αὐτῶν καὶ τῶν συμβεβηκότων. τὸ μὲν γὰρ περιττὸν ἔσται καὶ τὸ ἄρτιον καὶ τὸ εὐθὺ καὶ τὸ καμπύλον, ἔτι δὲ ἀριθμὸς καὶ γραμμὴ καὶ σχῆμα, ἄνευ κινήσεως, σὰρξ δὲ καὶ ὀστοῦν καὶ ἄνθρωπος οὐκέτι, ἀλλὰ ταῦτα ὥσπερ ῥὶς σιμὴ ἀλλ'' οὐχ ὡς τὸ καμπύλον λέγεται.

[3] Cf. *Metaphysica*, XIII and XIV.

> stances—except in thought—nor can they exist somewhere apart from [sensible substances].[4]

The alternative—that the mathematicals are substances *coinciding with* the sensible things to which they belong—being no less tenable, Aristotle asks whether they might not exist, truly, but in some sense other than as substances: "for *to be* has many senses."[5]

Aristotle's insight at this point is that it is the Platonic *turn* in separation, the very reason why one would say that mathematicals are really separate in the first place, that leads us into error. Just as claim-separations in thinking about natural things lead us to the Forms, claim-separations in mathematics lead us to Platonic mathematicals—substances occupying an immaterial realm. A *separation* consisting in setting the *per accidens* aside, however, has no such consequence:

> Just as we make many claims about things being in motion simply *as such*, apart from what such things and their accidents are, and yet it is not therefore necessary that there be either something in motion really separate from sensible substances or some [other] nature, in motion, in those very substances, so do we make claims and develop sciences about things in motion, though not *insofar as* they are in motion but only *as* bodies. So it is, too, with [thinking about them] only *as* planes, lines, divisibles, indivisibles having position, or indivisibles.[6]

In his logical work, *On Interpretation*, Aristotle distinguishes between two kinds of thinking:

[4] Ibid., XIII 31077b11–14. ὅτι μὲν οὖν οὔτε οὐσίαι μᾶλλον τῶν σωμάτων εἰσὶν οὔτε πρότερα τῷ εἶναι τῶν αἰσθητῶν ἀλλὰ τῷ λόγῳ μόνον, οὔτε κεχωρισμένα που εἶναι δυνατόν, εἴρηται ἱκανῶς:

[5] Ibid., 17. πολλαχῶς γὰρ τὸ εἶναι λέγομεν.

[6] Ibid., 1077b23–31. ὥσπερ γὰρ καὶ ᾗ κινούμενα μόνον πολλοὶ λόγοι εἰσί, χωρὶς τοῦ τί ἕκαστόν ἐστι τῶν τοιούτων καὶ τῶν συμβεβηκότων αὐτοῖς, καὶ οὐκ ἀνάγκη διὰ ταῦτα ἢ κεχωρισμένον τι εἶναι κινούμενον τῶν αἰσθητῶν ἢ ἐν τούτοις τινὰ φύσιν εἶναι ἀφωρισμένην, οὕτω καὶ ἐπὶ τῶν κινουμένων ἔσονται λόγοι καὶ ἐπιστῆμαι, οὐχ ᾗ κινούμενα δὲ ἀλλ᾽ ᾗ σώματα μόνον, καὶ πάλιν ᾗ ἐπίπεδα μόνον καὶ ᾗ μήκη μόνον, καὶ ᾗ διαιρετὰ καὶ ᾗ ἀδιαίρετα ἔχοντα δὲ θέσιν καὶ ᾗ ἀδιαίρετα μόνον. Emphases mine. Note Aristotle's examples of mathematical principles: surfaces, lengths, discrete or continuous quantity in general (that is, the divisible), points, and units.

> Just as in the soul there is thought without [it being] true or false and there is thought which must already be so, so, too, with speech. For the false and the true are about composition and separation.[7]

One way to understand Aristotle's observation here is to distinguish between separation among thoughts of this first sort and separation among thoughts of the second. To separate the simple thought of *circle* from the simple thought of *mutable* entails no falsehood—one simply considers the one while ignoring what is accidental to it. To separate *circle* from *mutable* in the second way is to make a claim: *circles are immutable*.[8] It is a separation of this latter sort which constitutes the Platonic turn and which earns the name *separation* in the passage cited. Being a separation of a fundamentally different sort, it has become customary to call thinking things separately in the first way, *abstraction*. Not bearing upon their separate *existence* but only upon what is *per se*, acts of abstraction leave a *real* account of the mathematicals, as accidents of physical substances, intact. *Episteme* is only of what is real, but the mathematicals *are* real, just not real substances existing in separation from sensible ones.

> Since it is simply true to say that both separable and inseparable things exist (there are things in motion), it is also simply true to say that the mathematicals exist, with such qualities as they are said to have.[9]

Nor does this *separation from what is accidental* apply to the mathematical sciences alone, even if *this particular form* of it does.[10]

[7] Aristotle, *De Interpretatione* (Oxford: Oxford University Press, 1936), 1, 16a9–11, hereafter *De Interpretatione*.

[8] This is why we have called these *claim-separations*, above.

[9] Ibid., 1077b31–33. ὥστ' ἐπεὶ ἁπλῶς λέγειν ἀληθὲς μὴ μόνον τὰ χωριστὰ εἶναι ἀλλὰ καὶ τὰ μὴ χωριστά (οἷον κινούμενα εἶναι), καὶ τὰ μαθηματικὰ ὅτι ἔστιν ἁπλῶς ἀληθὲς εἰπεῖν, καὶ τοιαῦτά γε οἷα λέγουσιν.

[10] Mathematics is characterized by abstracting the properly quantitative aspects of sensible substances from those aspects which are accidental to them. As is commonly said, it makes no difference to mathematicians whether the lines or circles they think about have these sensible qualities or those. Other sciences abstract from the *per accidens* in other ways, for example, true and universal claims in the natural sciences—even claims about place and time—abstract from the concrete where-conditions and when-conditions under which the realities exhibiting these claims were concretely found.

> And just as it is simply true to say of the other sciences that they are about some [one] thing but not about what is accidental to it (e.g., if [a science] is about health it will not also be about the pale, if the healthy happens to be pale), rather, each [science] is about its own thing—health if it is about health, man if it is about man—so too is it with geometry. The mathematical sciences are not about sensible things just because what they *are* about happens to be sensible (even though they do not take them that way), nor, on the other hand, are they about things separate from sensible things.[11]

Said positively, *episteme* is a sort of thinking about things with an invariable focus upon what is *per se*. If we think in this way about some property of a thing, then upon that property alone (and not the subject in which it *happens* to inhere). If we think, in turn, about the attributes of that property, then upon those attributes and that in which they *properly* inhere (and not properties which are conceptually dissociated from them).

> Many [characteristics] belong to things in themselves and as such: for example, to animal simply as female or as male, even though there is no female or male apart from animals. So, too, [characteristics belong to things] simply as lengths or as surfaces.[12]

Episteme accounts for *male* or *female* proper, not *cow* or *horse*, or even *mammal* or *animal*, knowing all the while that *only such things are male or female*. So, too, for the mathematicals, for which *episteme* accounts through *precise* thinking about lengths, surfaces, mathematical solids, numbers, and *their* properties. Examples from the *middle* sciences drive the point home:

> The same argument can be made in the case of harmonics and optics. They do not regard their subjects insofar as they are sights or sounds, but as

[11] *Metaphysica*, XIII 31077b33–1078a5. καὶ ὥσπερ καὶ τὰς ἄλλας ἐπιστήμας ἁπλῶς ἀληθὲς εἰπεῖν τούτου εἶναι, οὐχὶ τοῦ συμβεβηκότος (οἷον ὅτι λευκοῦ, εἰ τὸ ὑγιεινὸν λευκόν, ἡ δ᾽ ἔστιν ὑγιεινοῦ) ἀλλ᾽ ἐκείνου οὗ ἐστὶν ἑκάστη, εἰ ᾗ ὑγιεινὸν ὑγιεινοῦ, εἰ δ᾽ ᾗ ἄνθρωπος ἀνθρώπου, οὕτω καὶ τὴν γεωμετρίαν: οὐκ εἰ συμβέβηκεν αἰσθητὰ εἶναι ὧν ἐστί, μὴ ἔστι δὲ ᾗ αἰσθητά, οὐ τῶν αἰσθητῶν ἔσονται αἱ μαθηματικαὶ ἐπιστῆμαι, οὐ μέντοι οὐσὲ παρὰ ταῦτα ἄλλων κεχωρισμένων.

[12] Ibid., 1078a5–8. πολλὰ δὲ συμβέβηκε καθ᾽ αὑτὰ τοῖς πράγμασιν ᾗ ἕκαστον ὑπάρχει τῶν τοιούτων, ἐπεὶ καὶ ᾗ θῆλυ τὸ ζῷον καὶ ᾗ ἄρρεν, ἴδια πάθη ἔστιν (καίτοι οὐκ ἔστι τι θῆλυ οὐδ᾽ ἄρρεν κεχωρισμένον τῶν ζῴων): ὥστε καὶ ᾗ μήκη μόνον καὶ ᾗ ἐπίπεδα.

> lengths and numbers: properly speaking the latter are attributes of the former. So, too, with mechanics.[13]

Abstraction makes *episteme* possible. Considering what thought about things would be *could we not* set aside what is *per accidens*, we fully agree with Aristotle:

> We note, first, that there can be no speculative knowledge of things which *are* accidentally, one sign of which is that no science—practical, productive, or speculative—bothers with it.[14]

Aristotle next provides us with a clear example of how this works in mathematics:

> It is best that one think about each thing in this way, separating what is not separate, as the arithmetician and the geometer do. For a man as man is one indivisible thing, and [the arithmetician] posited one indivisible thing, and then thought about whether anything belongs to a man as indivisible. The geometer thinks about him neither as a man nor as indivisible but as a [mathematical] solid. For clearly those things which would have been in him even if he had not been indivisible [can] be in him apart from this [indivisibility].[15]

Our own analysis of *Metaphysics* XIII 3 ends with a *segue* to the second of the remaining difficulties: "For this reason, then, geometers speak

[13] Ibid., 1078a14–16. ὁ δ᾽ αὐτὸς λόγος καὶ περὶ ἁρμονικῆς καὶ ὀπτικῆς: οὐδετέρα γὰρ ᾗ ὄψις ἢ ᾗ φωνὴ θεωρεῖ, ἀλλ᾽ ᾗ γραμμαὶ καὶ ἀριθμοί (οἰκεῖα μέντοι ταῦτα πάθη ἐκείνων), καὶ ἡ μηχανικὴ δὲ ὡσαύτως. Note Aristotle's observation here regarding properties, which can be *hierarchical*.

[14] *Metaphysica*, VI 11026b2–6. πρῶτον περὶ τοῦ κατὰ συμβεβηκὸς λεκτέον, ὅτι οὐδεμία ἐστὶ περὶ αὐτὸ θεωρία. Σημεῖον δέ: οὐδεμιᾷ γὰρ ἐπιστήμῃ ἐπιμελὲς περὶ αὐτοῦ οὔτε πρακτικῇ οὔτε ποιητικῇ οὔτε θεωρητικῇ.

[15] Ibid., XIII 31078a21–29. ἄριστα δ᾽ ἂν οὕτω θεωρηθείη ἕκαστον, εἴ τις τὸ μὴ κεχωρισμένον θείη χωρίσας, ὅπερ ὁ ἀριθμητικὸς ποιεῖ καὶ ὁ γεωμέτρης. ἓν μὲν γὰρ καὶ ἀδιαίρετον ὁ ἄνθρωπος ᾗ ἄνθρωπος: ὁ δ᾽ ἔθετο ἓν ἀδιαίρετον, εἶτ᾽ ἐθεώρησεν εἴ τι τῷ ἀνθρώπῳ συμβέβηκεν ᾗ ἀδιαίρετος. ὁ δὲ γεωμέτρης οὔθ᾽ ᾗ ἄνθρωπος οὔθ᾽ ᾗ ἀδιαίρετος ἀλλ᾽ ᾗ στερεόν. ἃ γὰρ κἂν εἰ μή που ἦν ἀδιαίρετος ὑπῆρχεν αὐτῷ, δῆλον ὅτι καὶ ἄνευ τούτων ἐνδέχεται αὐτῷ ὑπάρχειν [τὸ δυνατόν].

rightly and discourse about existing things. Such things *do* exist: for being is twofold—things *are* actually and things *are* materially."[16]

Abstraction (as here described) distinguishes Aristotle's philosophy of mathematics from platonic realism in that he qualifies the kind of separation which occurs—it is a separation *in thought*.[17] The mathematicals *have* matter, but we ignore that about them. But can we ignore matter altogether? Certainly, Aristotle's allusion to matter here has *something to* do with the physical substances in which the mathematicals are found. On the other hand, do we not set such physicalities aside when we abstract from the *per accidens*? Do we not admit that they exist, in fact, in material substances but are defined (and so thought) apart from them? It turns out that, for the same reason Plato's mathematicals are said to be *between* the Forms and singular things, we cannot think about the mathematicals apart from so-called *intelligible* matter.

Intelligible Matter

In our discussion of *episteme*, above, we made reference to an objection Meno raises to Socrates' pursuit of the nature of moral virtue and to his even inquiring about whether virtue can be taught. Plato's *Meno* supplies us with an especially clear example of the Socratic *elenchus*, with the eponymous son of Alexidemus serving the role of the hapless victim. As we follow their discussion of the nature of moral virtue, we see Meno's definitions fall to Socrates' objections, one after another. But Meno is not so defenseless as might first appear. In his exasperation, and having insulted Socrates by comparing him (in appearance and in effect) to a torpedo-fish, he sets up an argument designed to stop Socrates' inquiry in its tracks:

> Meno: How will you inquire, Socrates, into that about which you do not know at all what it is? What sort of thing will you be looking for among the things you do not know? And even if you were to find it, how can you know that this is the thing which you did not know?

[16] Ibid., 1078a30–31. ὥστε διὰ τοῦτο ὀρθῶς οἱ γεωμέτραι λέγουσι, καὶ περὶ ὄντων διαλέγονται, καὶ ὄντα ἐστίν: διττὸν γὰρ τὸ ὄν, τὸ μὲν ἐντελεχείᾳ τὸ δ ὑλικῶς. Emphases mine.

[17] It differs, too, from *nominalism* in regarding mathematicals as known essences, existing in physical substances (nominalism presents itself in two forms: there are no real essences, or no real essences are known). Nominalism is one pathway to mathematical anti-realism.

Socrates: I know what you are trying to say, Meno. You do realize what a contentious argument you are introducing, that a man can inquire neither into what he knows nor into what he does not know: for he inquires either into what he knows, and so has no need to inquire, or into what he does not know, and is looking for he knows not what.[18]

Socrates' solution to the dilemma is his familiar account of recollection—we never learn things anew, but recollect things we have forgotten. Delighted with this response, Meno permits Socrates to continue their investigation into the nature of virtue.

But Socrates' solution is not real, merely an ad hoc response sufficient only to satisfy the eager young man. For, even in his account of recollection, Socrates assumes that *learning does occur*:

Socrates: Now, as the soul is immortal and has been born many times over and, *having seen all things* in this world and in the world below, she knows them all, it is no wonder that she can remember all that she knows about virtue and everything else she had *once come to know*. For, as the whole of nature is alike, nothing prevents her, in calling even one thing to mind (which men call 'learning'), from discovering all the rest. For all inquiry and all learning are recollection. Therefore we ought not to be persuaded by your contentious argument: for it will make us lazy and is pleasant only to soft men. But the other way will

[18] *Meno*, 80d-e. Μένων—καὶ τίνα τρόπον ζητήσεις, ὦ Σώκρατες, τοῦτο ὃ μὴ οἶσθα τὸ παράπαν ὅτι ἐστίν; ποῖον γὰρ ὧν οὐκ οἶσθα προθέμενος ζητήσεις; ἢ εἰ καὶ ὅτι μάλιστα ἐντύχοις αὐτῷ, πῶς εἴσῃ ὅτι τοῦτό ἐστιν ὃ σὺ οὐκ ᾔδησθα; Σωκράτης—μανθάνω οἷον βούλει λέγειν, ὦ Μένων. ὁρᾷς τοῦτον ὡς ἐριστικὸν λόγον κατάγεις, ὡς οὐκ ἄρα ἔστιν ζητεῖν ἀνθρώπῳ οὔτε ὃ οἶδε οὔτε ὃ μὴ οἶδε; οὔτε γὰρ ἂν ὅ γε οἶδεν ζητοῖ—οἶδεν γάρ, καὶ οὐδὲν δεῖ τῷ γε τοιούτῳ ζητήσεως—οὔτε ὃ μὴ οἶδεν—οὐδὲ γὰρ οἶδεν ὅτι ζητήσει. Meno's argument is a destructive dilemma. Either one knows or one does not. If one already knows, one cannot inquire (one cannot ask about what one already knows). If one does not know, one cannot inquire (as one will not be able to recognize the answer to the question). Thus, whether one knows or not, one cannot inquire.

> make us diligent and inquisitive. Consequently, trusting in that truth, I am willing to inquire with you into what virtue is.[19]

In *Posterior Analytics*, Aristotle considers and directly resolves this same dilemma, a real difficulty for the man who claims that "all rational teaching and learning come about through preexistent knowledge."[20] Like Socrates' solution, satisfactory to Meno but unsatisfactory in itself, Aristotle's claim might appear only to defer the problem: if all our knowledge comes from preexistent knowledge, then how did we acquire any in the first place? Aristotle does not get far into the work without tackling the dilemma head-on:

> One can come to know either through prior knowledge or also simultaneously with knowledge of some other thing, as, for example, one knows the particulars falling under a known universal.[21]

Notice that even here, at the very outset, Aristotle insists upon distinguishing among *types* of knowledge. He goes on to supply an example:

> One already knows that the angles of every triangle are equal to two right angles; but one comes to know [that the angles are equal to two right angles and] that the figure in the semicircle is a triangle at the same time.[22]

[19] Plato, "Meno", 81c-e. ἅτε οὖν ἡ ψυχὴ ἀθάνατός τε οὖσα καὶ πολλάκις γεγονυῖα, καὶ ἑωρακυῖα καὶ τὰ ἐνθάδε καὶ τὰ ἐν Ἅιδου καὶ πάντα χρήματα, οὐκ ἔστιν ὅτι οὐ μεμάθηκεν: ὥστε οὐδὲν θαυμαστὸν καὶ περὶ ἀρετῆς καὶ περὶ ἄλλων οἷόν τ᾽ εἶναι αὐτὴν ἀναμνησθῆναι, ἅ γε καὶ πρότερον ἠπίστατο. ἅτε γὰρ τῆς φύσεως ἁπάσης συγγενοῦς οὔσης, καὶ μεμαθηκυίας τῆς ψυχῆς ἅπαντα, οὐδὲν κωλύει ἓν μόνον ἀναμνησθέντα—ὃ δὴ μάθησιν καλοῦσιν ἄνθρωποι—τἆλλα πάντα αὐτὸν ἀνευρεῖν, ἐάν τις ἀνδρεῖος ᾖ καὶ μὴ ἀποκάμνῃ ζητῶν: τὸ γὰρ ζητεῖν ἄρα καὶ τὸ μανθάνειν ἀνάμνησις ὅλον ἐστίν. οὔκουν δεῖ πείθεσθαι τούτῳ τῷ ἐριστικῷ λόγῳ: οὗτος μὲν γὰρ ἂν ἡμᾶς ἀργοὺς ποιήσειεν καὶ ἥδιστα τοῖς μαλακοῖς τῶν ἀνθρώπων ἡδὺς ἀκοῦσαι, ὅδε δὲ ἐργατικούς τε καὶ ζητητικοὺς ποιεῖ: ᾧ ἐγὼ πιστεύων ἀληθεῖ εἶναι ἐθέλω μετὰ σοῦ ζητεῖν ἀρετὴ ὅτι ἐστίν. Emphasis mine.

[20] *Analytica Posteriora*, I.1.71a1–2. Πᾶσα διδασκαλία καὶ πᾶσα μάθησις διανοητικὴ ἐκ προϋπαρχούσης γίνεται γνώσεως.

[21] Ibid., 71a17–19. Ἔστι δὲ γνωρίζειν τὰ μὲν πρότερον γνωρίσαντα, τῶν δὲ καὶ ἅμα λαμβάνοντα τὴν γνῶσιν, οἷον ὅσα τυγχάνει ὄντα ὑπὸ τὸ καθόλου οὗ ἔχει τὴν γνῶσιν.

[22] Ibid., 71a19–21. ὅτι μὲν γὰρ πᾶν τρίγωνον ἔχει δυσὶν ὀρθαῖς ἴσας, προῄδει· ὅτι δὲ τόδε τὸ ἐν τῷ ἡμικυκλίῳ τρίγωνόν ἐστιν, ἅμα ἐπαγόμενος ἐγνώρισεν.

Rejecting the claim that, in knowing universals we know only, and ever, *incompletely*,[23] Aristotle is left with the conclusion that one can know, and simultaneously not know, the same thing at the same time—impossible unless one means something different by *know* in each case:

> Before *one is led to recognize this* or before *one reaches the conclusion*, we ought to say that in a way he knew and in a way he did not know. For if he simply did not know that it was [a triangle], how could he simply know that it had [angles equal to] two right angles? No. Clearly he knows, not simply, but universally.[24]

At this point, Aristotle immediately connects the alternative (that *know* can only mean one thing) to the dilemma proposed by Meno: "Otherwise, the problem found in the Meno comes about: either one learns nothing or what he [already] knows."[25]

Aristotle's solution distinguishes ways in which something can be known and foreknown:

> I think nothing prevents one from, on the one hand, knowing, and on the other, not knowing, what he is learning; for it would be absurd, not if he knew what he was learning in some way, but if he knew it in the very same way as he was learning it.[26]

To know universally is not to know each particular *simply*, or *as such*, as one can know that every pair is even without explicitly knowing all such pairs, assuming there is a pair of which one is unaware.

[23] Aristotle explicitly considers and rejects this position at *Analytica Posteriora*, I 1 71a30–71b5.

[24] Ibid., 71a24–39. πρὶν δ' ἐπαχθῆναι ἢ λαβεῖν συλλογισμὸν τρόπον μέν τινα ἴσως φατέον ἐπίστασθαι, τρόπον δ' ἄλλον οὔ. ὃ γὰρ μὴ ᾔδει εἰ ἔστιν ἁπλῶς, τοῦτο πῶς ᾔδει ὅτι δύο ὀρθὰς ἔχει ἁπλῶς; ἀλλὰ δῆλον ὡς ᾠδὶ μὲν ἐπίσταται, ὅτι καθόλου ἐπίσταται, ἁπλῶς δ' οὐκ ἐπίσταται. Emphasis mine. Aristotle is considering two instances of learning in this section: recognizing a singular as falling under a universal and arriving at a syllogistic conclusion. Though the two are associated, he spends more time discussing the former, while our own point more directly concerns the latter. Still, each type is comprised under what Aristotle claims for learning here.

[25] Ibid., 71a29–30. εἰ δὲ μή, τὸ ἐν τῷ Μένωνι ἀπόρημα συμβήσεται· ἢ γὰρ οὐδὲν μαθήσεται ἢ ἃ οἶδεν.

[26] Ibid., 71b5–8. ἀλλ' οὐδέν (οἶμαι) κωλύει, ὃ μανθάνει, ἔστιν ὡς ἐπίστασθαι, ἔστι δ' ὡς ἀγνοεῖν· ἄτοπον γὰρ οὐκ εἰ οἶδέ πως ὃ μανθάνει, ἀλλ' εἰ ᾠδί, οἷον ᾗ μανθάνει καὶ ὥς.

Consider Aristotle's example of the figure in a semicircle. That *all triangles have their angles equal to two right angles* is actually known as a premise of the argument, while what is as yet not actually known is whether the figure inscribed in the semicircle does. There are two possibilities: *the figure in question has its angles equal to two right angles* or *the figure in question does not have its angles equal to two right angles.* But even this is not quite true. We all know that the figure inscribed in a semicircle *is* a triangle and so, in fact, does have its angles equal to two right angles. It turns out that *the figure in question does not have its angles equal to two right angles* is not even *possibly* true. On the other hand, if it were not at least possible, why would we ask the question in the first place? That is because the merely logical possibilities are *potential.* Matter is in potency to any number of forms, and the *matter* of this proposition (its *terms*) can be arranged in the form of an affirmation or a denial: nothing prevents them being so formulated. Nevertheless, only one *conclusion* follows, given the principles. Hence, it is possible not to know something which is in fact so, and possible not to have demonstrated something which is demonstrable. All the same, what has not yet been proven so, is so.[27]

In Chap. 6 we will return to this distinction between knowing something potentially—or *virtually*—and knowing it actually as we see it developed in Aquinas. That distinction is founded upon the above solution to Meno's dilemma, that we know particulars and conclusions *differently* at the outset and in conclusion. Further, though Aristotle does not explicitly refer to actual, virtual, or potential knowledge in these passages, the solution serves to explicate the passage which led to this section: "[G]eometers speak rightly and discourse about existing things. Such things *do* exist: for being is twofold—things *are* actually and things *are* materially."[28]

[27] Failure to make this distinction led intuitionist mathematicians of the twentieth century to deny the universality of the principle of excluded middle. This observation parallels a famous distinction made by Thomas Aquinas in *Summa Theologiae,* I 2 1 between propositions known *per se* and *quoad nos* and those known *per se* but not *quoad nos.* Propositions are known *per se* whose predicate is included in the *ratio* of the subject; still, it is possible we may not have seen this, as we may be ignorant of the *ratio* of the subject. So, too, with the conclusions of demonstrations which, in contrast, are known *per aliud* and not *per se*: a proposition may, in fact, follow necessarily from principles known *per se*, even if no one has come to see the inference. They would be demonstrated *in themselves*, but not *to us.*

[28] *Metaphysica* XIII 3, 1078a30–31. ὥστε διὰ τοῦτο ὀρθῶς οἱ γεωμέτραι λέγουσι, καὶ περὶ ὄντων διαλέγονται, καὶ ὄντα ἐστίν: διττὸν γὰρ τὸ ὄν, τὸ μὲν ἐντελεχείᾳ τὸ δ ὑλικῶς. Emphases mine.

Intelligible Matter and the Reality of Mathematicals

One place Aristotle takes up being *potentially*—being *materially*—can be found in *Metaphysics* VII where, in chapters ten and eleven, he returns to the mathematicals, among other things. What has being *potentially*—or *materially*—to do with mathematics? Firstly, as we have seen, it is relevant to the dilemma we addressed above, as one does not actually know a conclusion (whether mathematical or otherwise) at the outset, though one does—must—know it in some way: potentially, or virtually. So much for mathematical *reasoning*. As to the mathematicals themselves, it seems strange to speak of them as having matter—their *physicality* would appear to be the very thing from which we abstract when thinking about them. On the other hand, there is a sort of matter here as well.[29]

While Book VII of the *Metaphysics* is generally about substance, one sense of substance is the *substratum*—*τὸ ὑποκείμενον*—and, in its first sense, the *substratum* is matter.[30] To what extent does a consideration of matter enter into our thought about things, especially when it takes the form of *episteme*? If we recall that one of the conditions for *episteme* is that we already know *what* the subject of our conclusion is—its real definition—then matter is the concern of the mathematical sciences, too, to the extent that it enters into the definition of the mathematicals.

In chapters ten and eleven, Aristotle raises a question whose solution sheds much light on his mathematical philosophy:

> Since a definition is an expression [λόγος] and every such λόγος has parts, and the λόγος is related to the thing in the same way as part of the λόγος is to part of the thing, there is a question whether the λόγος of the parts is in the λόγος of the whole or not.[31]

[29] As we mentioned above, Aristotle does speak of *intelligible* matter (ὕλη νοητὴ) in several places. Even in logic, we say that one part of a definition is material in relation to another part, which is formal.

[30] Cf. *Metaphysica,* VII 31028b33–1029a2.

[31] *Metaphysica* VII 101034b20–24. ἐπεὶ δὲ ὁ ὁρισμὸς λόγος ἐστί, πᾶς δὲ λόγος μέρη ἔχει, ὡς δὲ ὁ λόγος πρὸς τὸ πρᾶγμα, καὶ τὸ μέρος τοῦ λόγου πρὸς τὸ μέρος τοῦ πράγματος ὁμοίως ἔχει, ἀπορεῖται ἤδη πότερον δεῖ τὸν τῶν μερῶν λόγον ἐνυπάρχειν ἐν τῷ τοῦ ὅλου λόγῳ ἢ οὔ. Λόγος is notoriously difficult to translate. Here, it is the intellectual expression of the nature of a thing.

Aristotle sees in the relation of part to whole a reference to matter: parts are *materially* in a whole. But both the mathematicals themselves and their definitions contain parts—indeed, being composite expressions, *all* definitions contain parts. In fact, we earlier observed that mathematics is especially concerned with parts, after all, parts are of the essence of quantity. If parts are *materially* in a whole, then, and mathematics is a science of parts—or things having parts—matter will need to be included in the definition of the mathematicals in some way. His question, here, is *in what way*? He continues by saying:

> For sometimes [the λόγος of the part] appears in [the λόγος of the whole] and at other times it does not. The λόγος of the circle does not contain [the λόγος] of the segments, whereas [the λόγος] of the syllable does contain [that] of the letters: and yet the circle is divided into segments, just as the syllable is divided into letters."[32]

This difference comes down to a difference among types of parts—there are parts of things and parts of the λόγος of things. We think about such parts differently, as it turns out:

> Parts which are like matter and into which [the whole] is divided as into matter are posterior [to the whole], whereas parts of the λόγος and of the essence expressed in that λόγος are prior [to the whole]—whether all or some of them.[33]

Much to our purpose here, Aristotle supplies an instance of each taken from mathematics. Though a circle can be (literally) divided into two parts—semicircles—it is not in knowing these parts that one arrives at a knowledge of the circle, its definition. Knowing such parts is *subsequent to* knowing what circles are, not vice versa. On the other hand, the *definition* of the semicircle includes the notion of *circle* among *its* parts—since a semicircle is *half a circle*—with the result that we know what semicircles are *after* we know circles, not vice versa. *Circle* is *part* of *semicircle*, then, not part of it, but part of its definition.

[32] Ibid., 1034b24–27. ἐπ ἐνίων μὲν γὰρ φαίνονται ἐνόντες ἐνίων δ οὔ. τοῦ μὲν [25] γὰρ κύκλου ὁ λόγος οὐκ ἔχει τὸν τῶν τμημάτων, ὁ δὲ τῆς συλλαβῆς ἔχει τὸν τῶν στοιχείων: καίτοι διαιρεῖται καὶ ὁ κύκλος εἰς τὰ τμήματα ὥσπερ καὶ ἡ συλλαβὴ εἰς τὰ στοιχεῖα.

[33] Ibid., 1035b11–13. ὥσθ ὅσα μὲν μέρη ὡς ὕλη καὶ εἰς ἃ διαιρεῖται ὡς ὕλην, ὕστερα: ὅσα δὲ ὡς τοῦ λόγου καὶ τῆς οὐσίας τῆς κατὰ τὸν λόγον, πρότερα ἢ πάντα ἢ ἔνια.

Recalling that the mathematicals, for Aristotle (and for Plato too, though differently) are *forms*, one might initially be led to think that (as above) matter is no part of their consideration. Indeed, when we think about them, we are careful to ignore any material aspect, as being accidental. On the other hand, definitions are composite wholes, themselves made up of parts, so the part-to-whole relationship still obtains and therefore *materiality*, of some sort. There follows Aristotle's direct reference to this type of matter:

> Now a part may be a part of the form (by *form* I mean the essence), or of the whole composed of form and matter, or of the matter alone. But only the parts of the form are parts of the λόγος—the λόγος of the universal. … Some matter is sensible and some is intelligible (ὕλη νοητή): sensible, for example bronze and wood and all changeable matter and intelligible, belonging to sensible things but not [intelligible] as sensible, for example the mathematicals.[34]

It is difficult, says Aristotle, to identify what sort of parts belong to the form and what do not.[35] Still, without such a distinction, we can know nothing scientifically, since a definition is the λόγος of the form and there is no *episteme* without definition. On the other hand, the λόγος of the form (as we have just seen) is not severed from materiality altogether—in fact, just the opposite, as it must include a reference to the parts of the form. This is as true of the mathematicals (which *abstract* from materiality—that is, *sensible* matter) as of other things. This last quotation brings us to Aristotle's conclusion:

> As for the mathematicals, why are the λόγοι of the [concrete] parts not parts of the λόγος of the [abstract] whole—why are the λόγοι of semicircles [not parts of the λόγος] of the circle? For they are not sensible. Maybe it makes no difference? Since there will be matter even of some things which are not sensible? For, indeed, everything which is not an essence, or its own form,

[34] Ibid., 1035b33–1036a1, 1036a9–13. μέρος μὲν οὖν ἐστὶ καὶ τοῦ εἴδους (εἶδος δὲ λέγω τὸ τί ἦν εἶναι) καὶ τοῦ συνόλου τοῦ ἐκ τοῦ εἴδους καὶ τῆς ὕλης καὶ τῆς ὕλης αὐτῆς. Ἀλλὰ τοῦ λόγου μέρη τὰ τοῦ εἴδους μόνον ἐστίν, ὁ δὲ λόγος ἐστὶ τοῦ καθόλου: … ὕλη δὲ ἡ μὲν αἰσθητή ἐστιν ἡ δὲ νοητή, αἰσθητὴ μὲν οἷον χαλκὸς καὶ ξύλον καὶ ὅση κινητὴ ὕλη, νοητὴ δὲ ἡ ἐν τοῖς αἰσθητοῖς ὑπάρχουσα μὴ ᾗ αἰσθητά, οἷον τὰ μαθηματικά.

[35] This question occupies Aristotle in *Metaphysica,* VII 11.

> in itself, but is a particular thing, has *some* matter. [Semicircles] will be not parts of the universal circle but of particular circles, as we said before—for some matter is sensible and some is intelligible.[36]

The mathematicals, by Aristotle's account, are accidental forms of material substances. Abstracting from what is *per accidens*, as we saw, we can think about such forms while ignoring those aspects of their *materiality* which has nothing to do with what *they* are. Contrarily, having material parts is of the essence of natural, living things—we cannot think about them while ignoring these. We *can* (and do) ignore their *individuality* when we define them and make universal claims about them. Still, these claims *include* their materiality, in general, as it were.

Were the mathematicals the sort of thing Plato envisioned, immaterial substances, though many the same, our thinking of them could (and necessarily would) set aside *any* notion of materiality. In Aristotle's terms, each would be a unique substantial form, which form would *constitute* its substance. But it is physical substances which have real quantitative aspects—our *how much* and *how many* questions occur when we think about *physical*—not immaterial—things. However little mathematics has to do with motion, rest, heavy, light, color, rough, and smooth, we cannot say there is *no* materiality, in any sense, in this realm. Physical substance itself is necessarily a condition for the very existence of the mathematicals. We cannot ignore—abstract from—*that* materiality, since *particular circles* do require matter to be. Moreover, as we pointed out earlier, thinking about quantity is to think about parts, or things with parts. It makes much difference whether the parts are formal or material, as Aristotle has noted, if we ask what must be known *prior* to what. Still, eventually *and* at the outset, mathematicians must deal with parts, and therefore matter.

This observation, important in itself as to the *beginnings* of mathematics, also has implications for *conducting* mathematics, as we will see in what follows.

[36] *Metaphysica* VII 111036b33–1037a4. περὶ δὲ τὰ μαθηματικὰ διὰ τί οὐκ εἰσὶ μέρη οἱ λόγοι τῶν λόγων, οἷον τοῦ κύκλου τὰ ἡμικύκλια; οὐ γάρ ἐστιν αἰσθητὰ ταῦτα. ἢ οὐθὲν διαφέρει; ἔσται γὰρ ὕλη ἐνίων καὶ μὴ αἰσθητῶν: καὶ παντὸς γὰρ ὕλη τις ἔστιν ὃ μὴ ἔστι τί ἦν εἶναι καὶ εἶδος αὐτὸ καθ᾽ αὑτὸ ἀλλὰ τόδε τι. κύκλου μὲν οὖν οὐκ ἔσται τοῦ καθόλου, τῶν δὲ καθ᾽ ἕκαστα ἔσται μέρη ταῦτα, ὥσπερ εἴρηται πρότερον: ἔστι γὰρ ὕλη ἡ μὲν αἰσθητὴ ἡ δὲ νοητή.

Intelligible Matter and Access

We have made much in the previous pages of Aristotle's insistence that the mathematicals be abstracted from their physical conditions when we think about them, but not all that is physical about them. This balance between the abstract and the material allows him to speak of them as *idealized* (through the act of abstraction) and yet *real*, each having an individual, physical subject in which it inheres and even universal definitions which do not prescind from *that* sort of materiality. That connection to the *physically real* explains why Aristotle was even able to raise the question whether mathematics and the natural sciences differ. It also allows Aristotle to identify the source of the mathematicals' *unity* (recall the distinction between what is *per se* one and a collection, or heap). As we have seen, *physical substance* is not accidental to mathematical forms, but rather a precondition of them.[37] While we can enumerate the mathematicals as "planes, lines, divisibles, indivisibles having position, or indivisibles,"[38] using a kind of shorthand, we cannot forget that these are divisible or indivisible *physical substances: substance according as it is subject to quantity*, as Aquinas says.[39] The *realism* of Aristotle's mathematical realism is grounded in the reality of individual physical substances.

Why, finally, would the method of mathematics include *construction*, then?[40] *That* it does is attested to by everyone, Aristotle included, but, on the face of it, the need for construction seems incompatible with the need for abstraction—if we abstract, we have no need to construct, for physical circles, lines, planes, and numbers are already there.

In Book Nine of the *Metaphysics*, Aristotle considers one of his earlier distinctions of *being*: being actually and potentially. In line with the matter we have just addressed, he takes up the question of potentiality in mathematics, this time on the part not of what is *presupposed* to our thinking about the mathematicals (the parts of the form, which we must know

[37] Though much less than we see with the objects of the natural sciences, where we ignore matter to a different extent. For example, though neither this oxygen nor this hydrogen are relevant when we think of water, certainly hydrogen and oxygen are. In contrast, no type of physical substance is relevant when we think of circle, but physical substance in general—intelligible matter—is.

[38] *Metaphysica,* XIII 3, 1077b23–31

[39] *Summa Theologiae*, I 85 1 *ad* 2. "Materia vero intelligibilis dicitur substantia secundum quod subiacet quantitati."

[40] This is the third of the questions with which we were left at the end of the last chapter.

before we know their definitions) but on the part of *conducting mathematical demonstrations*, proving proper attributes of the ultimate subjects of mathematics. Constructability (even linguistically) implies potency of some sort. What we construct does not preexist our constructing it, unless potentially.

In *Metaphysics* IX 9, Aristotle turns to how potentiality is of concern to the mathematician—not, in this case, the matter *in which* the mathematicals themselves inhere, but to the potentials *we actualize* through constructions.

> [Geometrical] diagrams are discovered by means of an actualization, for we [discover] them by dividing. If the division were already made it would be clear, but as it is [the division] is there potentially. Why is the triangle [equal to] two right angles? Because the angles about one point are equal to two right angles. If the line [parallel] to the side had been set upward it would have been clearly seen at once. Why is the angle in a semicircle always right? If three [lines] are equal—two making the base and one set up straight from the middle—it is clearly seen by one who knows the former [proposition]. So it is evident that potential constructions are discovered by being made actual. This is because thinking [constructing them] is an actuality. Potentiality comes from actuality, then, and the ones making these constructions come to know through them. (For an act which is one in number is posterior in generation.)[41]

For Aristotle, not only are mathematical constructions needed *heuristically*, they are integral to mathematics as a type of *episteme*. Recalling what we said earlier regarding the conditions for *episteme*, a demonstration yields a conclusion in which a property is shown to belong to some subject, about which we already know what it is and that it exists.[42] As regards existence, then, Aristotle sees it as needing proof. One is not able to know of the existence of a thing in defining it (the first act of the mind) but only

[41] *Metaphysica*, IX 9 1051a21–34. εὑρίσκεται δὲ καὶ τὰ διαγράμματα ἐνεργείᾳ: διαιροῦντες γὰρ εὑρίσκουσιν. εἰ δ᾽ ἦν διῃρημένα, φανερὰ ἂν ἦν: νῦν δ᾽ ἐνυπάρχει δυνάμει. Διὰ τί δύο ὀρθαὶ τὸ τρίγωνον; ὅτι αἱ περὶ μίαν στιγμὴν γωνίαι ἴσαι δύο ὀρθαῖς. εἰ οὖν ἀνῆκτο ἡ παρὰ τὴν πλευράν, ἰδόντι ἂν ἦν εὐθὺς δῆλον διὰ τί. ἐν ἡμικυκλίῳ ὀρθὴ καθόλου διὰ τί; ἐὰν ἴσαι τρεῖς, ἥ τε βάσις δύο καὶ ἡ ἐκ μέσου ἐπισταθεῖσα ὀρθή, ἰδόντι δῆλον τῷ ἐκεῖνο εἰδότι. Ὥστε φανερὸν ὅτι τὰ δυνάμει ὄντα εἰς ἐνέργειαν ἀγόμενα εὑρίσκεται: αἴτιον δὲ ὅτι ἡ νόησις ἐνέργεια: ὥστ᾽ ἐξ ἐνεργείας ἡ δύναμις, καὶ διὰ τοῦτο ποιοῦντες γιγνώσκουσιν (ὕστερον γὰρ γενέσει ἡ ἐνέργεια ἡ κατ᾽ ἀριθμόν).

[42] Cf. *Analytica Posteriora*, I 1.

in its proof. This becomes clear in *Posterior Analytics* II 7, where Aristotle takes up the question of proving *that a thing is*—the being of truth.

> *That a thing is* will be [known through] a demonstration, which is exactly what the sciences do. Geometers assume what *triangle* means, but *that it is* they demonstrate.[43]

Constructions are mental acts which bring it about that we potentialities in the mathematicals brought to light. The angle in a semicircle being only right is something we came to see after never having seen it. Nor is this the more-or-less *open* potentiality of the various materials used in physical constructions: it is impossible to construct any *other* sort of triangle in that space. Constructions of this sort accord with the general conditions Aristotle lays out for *episteme*—that we know the proper reason why a necessary conclusion is true.

Using geometry as an example, exactly how do we even *begin* to think about one- and two-dimensional figures, lines and circles, for example? If we return to the very beginning of Euclid's *Elements* we learn that the existence of straight lines and circles is not proved in a standard demonstration but *postulated*, and such postulates *are* constructions, of a sort.[44] Given these, we can establish the existence of what are *commonly* thought to be the subjects of geometry: triangles, squares, and so on. But is this the *episteme* about which we have been speaking in this book? Has the activity of abstracting lines, circles, and other geometric figures already occurred? Perhaps. After all, Euclid does begin each of the Books with definitions of the geometricals which will be under consideration. On the other hand, those geometricals seem not to be the physical lines and circles we are looking at—even *after* we have left out of consideration those sensible attributes which are accidental to them. None of *those* lines are breadthless, none lie evenly with their points, and none of *those* surfaces lie evenly with the straight lines they contain. As we complete our consideration of his philosophy of mathematics, we must determine precisely what Aristotle *is* and *is not* saying in *Posterior Analytics*.

[43] *Analytica Posteriora*, 92b14–16. ἀπόδειξις ἄρ'' ἔσται ὅτι ἔστιν. Ὅπερ καὶ νῦν ποιοῦσιν αἱ ἐπιστῆμαι. τί μὲν γὰρ σημαίνει τὸ τρίγωνον, ἔλαβεν ὁ γεωμέτρης, ὅτι δ'' ἔστι, δείκνυσιν. Emphases mine.

[44] Cf. Euclid, *The Elements*, trans. Thomas L. Heath (New York: Dover Publications, 1956), I, Postulates 1–3.

Episteme requires that we already know some things: the truth of the premises of demonstration, the meaning of the term we use to signify some proper attribute we intend to prove about the subject, and the subject itself, both *that it is* and *what it is*. Aristotle's own example of something whose meaning alone we must already know, oddly enough, is *triangle*. Not some property *of* a triangle, rather triangle is regarded as a proper attribute to be proved as belonging to *another subject*.

> It is necessary to have preexistent knowledge of two sorts: for in some cases it is necessary to assume that something is, in others what the term is, and in still others both. For example, of *concerning everything, either the affirmation or the denial is true* [we assume] that it is, of *triangle* [we assume] what it means, and of *unit* [we assume] both what it means and that it is.[45]

Most unfortunately, Aristotle does not elaborate upon his examples. Moreover, the example is *mixed*—presumably there is no single demonstration proving that a unit is a triangle in light of the principle of excluded middle. Given what he says of it, though, *triangle* must be a proper attribute. But of what? Is *triangle* not the *subject* under consideration?

In his *Commentary on Aristotle's Posterior Analytics*, Aquinas provides us with a helpful insight as to Aristotle's meaning here:

> Since accidents are referred to substance in a certain order, it is not unreasonable that what is an accident with respect to one thing may also be a subject with respect to another. For example, surface is an accident of a bodily substance: yet surface is also the first subject of color.[46]

Common opinion tells us arithmetic is about numbers and that geometry is about plane and solid figures. Given an order among the accidents of a physical substance, however, triangle *might inhere* in a substance by

[45] *Analytica Posteriora*, I 1 71a11–16. διχῶς δ᾽ ἀναγκαῖον προγινῶσκειν· τὰ μὲν γάρ, ὅτι ἐστι, προϋπολαμβάνειν ἀναγκαῖον, τὰ δέ, τί τὸ λεγόμενόν ἐστι, ξυνιέναι δεῖ, τὰ δ᾽ ἄμφω, οἷον ὅτι μὲν ἅπαν ἢ φῆσαι ἢ ἀποφῆσαι ἀληθές, ὅτι ἐστι, τὸ δὲ τρίγωνον, ὅτι τοδὶ σημαίνει, τὴν δὲ μονάδα ἄμφω, καὶ τί σημαίνει καὶ ὅτι ἔστιν. It is in the course of his argument for this contention that we find Aristotle's answer to the above-mentioned *Meno dilemma*.

[46] Thomas Aquinas, *In Aristotelis Libros Posteriorum Analyticorum* Expositio (Rome: Marietti Editori Ltd., 1964), I 2 17, hereafter *In Posteriorum*. "Cum enim accidentia quodam ordine ad substantiam referantur, non est inconveniens id quod est accidens in respectu ad aliquid, esse etiam subiectum in respectu alterius. Sicut superficies est accidens substantiae corporalis: quae tamen superficies est primum subiectum coloris."

way of some more fundamental accident—its more *immediate* and not *ultimate* subject, so to speak. On the other hand, surely it is about *triangle* that we prove that its angles are equal to two right angles and about *right triangle* that we prove that the sum of the squares on each of the legs are equal in area to the square on its hypotenuse. *Triangle* and *right triangle* are undeniably the subjects of conclusions of demonstrations taking those forms. Aquinas continues to unravel Aristotle's thought:

> In those sciences which are about certain accidents, nothing prevents that which is taken as a subject with respect to a certain attribute to be also taken as an attribute with respect to a prior subject. Still, this cannot go on infinitely: for one must come to something which is first in that science, which is taken as a subject and in no way as an attribute; as is clear in the mathematical sciences, which are about continuous or discrete quantity. *For those things which are first in the genus of quantity are assumed in these sciences: for example, the unit, and the line and surface, and others of this sort.* Assuming these, certain other things are sought through demonstration, such as the equilateral triangle and the square in geometry, and other such things. This is why such demonstrations are called *operative*, as it were, for example: on a given line to construct an equilateral triangle. This achieved, certain attributes are then proved about it, for example, that its angles are equal, or some such thing. It is clear, therefore, that in the first manner of demonstration *triangle* is an attribute, while in the second it is a subject. So the Philosopher is using *triangle* as an attribute here, and not as a subject, when he says "of *triangle* it is necessary to already know what it means."[47]

[47] Ibid. Emphasis mine. "In illis autem scientiis, quae sunt de aliquibus accidentibus, nihil prohibet id, quod accipitur ut subiectum respectu alicuius passionis, accipi etiam ut passionem respectu anterioris subiecti. Hoc tamen non in infinitum procedit. Est enim devinire ad aliquod primum in scientia illa, quod ita accipitur ut subiectum, quod nullo modo ut passio; sicut patet in mathematicis scientiis, quae sunt de quantitate continua vel discreta. Supponuntur enim in his scientiis ea quae sunt prima in genere quantitatis; sicut unitas, et linea, et superficies, et alia huiusmodi. Quibus suppositis, per demonstrationem quaeruntur quaedam alia, sicut triangulus aequilaterus, quadratum in geometricis et alia huiusmodi. Quare quidem demonstrationes quasi operativae dicuntur, ut est illud, Super rectam lineam datam triangulum aequilaterum constituere. Quo adinvento, rursus de eo aliquae passiones probantur, sicut quod eius anguli sunt aequales aut aliquid huiusmodi. Patet igitur quod triangulus in primo modo demonstrationis se habet ut passio, in secundo se habet ut subiectum. Unde Philosophus hic exemplificat de triangulo ut est passio, non ut est subiectum, cum dicit quod de triangulo oportet praescire quoniam hoc significat."

While we will have more to say about the distinction between ultimate and proximate subjects at the end of Chap. 7, what is clear at least is that mathematical subjects are, in part, *made*. As Charles DeKoninck says:

> To show, concerning the equilateral triangle, that it is, it is not sufficient to point to a figure on the blackboard, so carefully drawn that to our eyes its three sides are indistinguishable in length; for no amount of physical measurement could verify the exactness of 'equal sides.' … While the geometer assumes the continuum as 'what is divisible without end,' according to one or more dimensions, any subject of which he demonstrates some property, e.g. of 'triangle,' must first be established by way of a construction to show that there is such a thing. Demonstrations by way of construction are called 'quasi operational.'[48]

A triangle is an attribute of a more fundamental subject, plane surface. As we develop the science of geometry, we need initially know only what *triangle* means. Knowing this, we recognize that the figure we have constructed is, in fact, a triangle, that is, *that triangles exist*. This constructive demonstration follows from the first principles of the science, including the postulates—we know that such principles are true. To say that geometry has, as its subject, triangles and squares and so on is not *false*, then, for these *are* subjects with respect to the proper attributes we subsequently demonstrate of them. Still, there is a *prior* subject: continuous quantity, according to one, two, or three dimensions. *This* is the subject we know through abstraction and about which we demonstrate, by way of construction, the proper attributes which *constitute* the different geometrical figures.

It is that about which we must already know both *that it is* and *what it is*, which is the ultimate subject of *episteme*.[49] For unless one already knows both what his subject is and that it exists, there is no possibility of demonstration—Aristotle has made this most clear. In fact, this very condition for *episteme* is what compelled us to ask whether the mathematicals exist as attributes of physical substances: for we realized that without our knowing

[48] Charles DeKoninck "Abstraction From Matter," in *Laval Théologique et Philosophique*, 13 no. 2 (1957): 133–96.

[49] Assuming, from above, what we said regarding intelligible matter, in their *long forms*, we could describe the ultimate subjects of arithmetic and geometry, respectively, as being *(quantitatively) indivisible physical substance*, and *(quantitatively) divisible physical substance*, respectively—*the unit* and *continuous quantity*, in brief.

of the existence of the subjects of the science (as we supposed them to be) there could be no science at all.

In sum, following Aristotle, we know that proper attributes of the subject of the mathematical sciences—the mathematicals and *their* properties—are real and necessarily follow from those subjects by way of construction.

As to details, Aristotle's own example of the subject of *episteme*—the unit—serves our purposes well. Nothing is more fundamental in arithmetic than the unit. It is the principle of number itself, for a number *is* a plurality of units, and any number can be constructed from it. In fact, arithmetic does not have difficulties of the same sort as geometry in this regard: for there appears to be no difference between the physical unit and the imagined one, as such, quite unlike the physical sphere and the imagined one.[50] The discrepancy between what we defined as a line or circle above and those we find as the concrete limits of natural substances is a sure sign that we do not abstract those forms from natural things, but, as we have just seen, construct them as proper attributes of the mathematical *continua* we *have* abstracted.

Aristotle's philosophy of mathematics aligns perfectly with the idea that numbers can be found in physical substances.[51] This claim seemed to justify arithmetic's designation as *episteme* from the beginning. On the other hand, if the above account of mathematical construction is also Aristotelian, all that seems to be required *in re* is the unit. Earlier—even apart from the precision problem in geometry—we struggled with the *believability* of physical numbers in arithmetic. Honestly, there seems to be no *need* for them. We, in fact, do not consult our experience to discover a physical substance with twenty-eight parts, abstract that number from its concrete conditions, and only *then* demonstrate that twenty-eight is a perfect number. The requirement becomes more obviously absurd when we move to higher orders of number. Seeing that the number one million is a square requires far less than searching out a substance with so great a number of parts, individually discernable.

Taking Aristotle as a whole, then, it seems that the reality of the mathematical unit and of continuous quantity in one, two, and three

[50] That is, the precision problem seems less of a problem for arithmetic, at the very least, than for geometry.

[51] This goes all the way back to Aristotle's including discrete quantities among the categories.

dimensions is sufficient for the mathematical sciences. This account resolves the difficulties which arose in dealing with the mathematicals (numbers, lines, triangles, dodecahedrons) and explains why we might consider them what mathematics is about in the first place. They are, after all, subjects in those sciences—albeit not the *ultimate* subjects. As to whether this undermines Aristotle's claim to realism in mathematics, much depends upon the nature of mathematical constructions (After all, one of the chief objections to mathematical realism is that the mathematicals are mere constructions, arbitrary, conventional.)

Earlier, we distinguished the kind of construction to which Aristotle makes explicit reference from the *openness* of the constructability of physical matter. Importantly, in saying that the constructing mind *actualizes* the potentiality in intelligible matter, Aristotle does not regard this potency as a fundamental one—like prime matter, for instance. While we will explore the distinction we find implicit in Aristotle's account of intelligible matter in the next chapter, we can say here that the potentiality of which he speaks is necessarily *limited*. This is at least clear from the example he gives—constructing a triangle in a semicircle, for it is not as if mathematical artisans can construct anything else in a semicircle. The potency here is restricted to a single outcome (as would be needed to fulfill the conditions for *episteme* anyway, since *episteme* is only of what must be.)

Having supplied a solid grounding in both the Platonic and Aristotelian accounts of the nature of the mathematicals and the sciences whereby we know them, we will now turn to the central portion of this work, a consideration of Thomas Aquinas' philosophy of mathematics. At issue are the ways in which Aquinas' thought aligns with, and departs from, his intellectual forebears. It is our contention that his view is an organic development of ancient mathematical wisdom. We also hold that these developments—and their foundations in Greek thought—help traditional philosophers to avoid some of the difficulties modern philosophers have encountered in thinking about mathematical things.

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PART II

Mathematical Realism in Aquinas

CHAPTER 6

Objects, Freedom, and Art

Mathematics and Existence

One finds in Aquinas an account of the mathematicals which parallels Aristotle's—even in distinguishing itself from Plato's view. The mathematicals are real attributes of physical substances about which we think by way of abstraction. On the basis of principles we already know and in conjunction with the other conditions laid out for *episteme* (Latin *scientia*), we demonstrate true and universal conclusions about continuous and discrete quantity by way of constructive proofs. In fact, as one reads Aquinas' account, it is difficult to find a mathematical point about which the two philosophers *disagree*—which is not to say that the latter did not build upon what was only dealt with *in principle* in the Stagirite's mathematical thought.

One possible difference, bearing upon the manner in which the mathematicals *exist,* arises from a *disputed question* Aquinas had apparently inserted into his *Commentary on the Sentences of Peter Lombard*.[1] He begins by noting that concepts in our minds can be related to external realities in three ways.

[1] See Armand Maurer, "A Neglected Thomistic Text on the Foundation of Mathematics," *Medieval Studies* 21 (1959): 185–92 See also Armand Maurer, "Thomists and Thomas Aquinas on the Foundation of Mathematics," *Review of Metaphysics* 47 (1993): 43–61 and Antoine Dondaine, "Saint Thomas et La Dispute des Attributs Divins (I Sent. d. 2, a. 3): Authenticité et Origine," *Archivum Fratrum Praedicatorum* 8 (1938): 253–62.

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> Sometimes the intellect conceives the likeness of a thing existing outside the soul (for example, what it conceives about the word *man*), and such a conception of the intellect has a foundation in the thing immediately, insofar as that very thing, through its conformity with the intellect, causes the intellect to be true and the name signifying that thing to be properly said of it.[2]

This is the standard Aristotelian and Thomistic account of intentional reception, in which, our mind being conformed to external reality, there results a formal *identity* between the mind and the thing known. Following what we have said of Aristotle's mathematical philosophy, above, and of Aquinas' agreement with it, we would expect our knowledge of the mathematicals to belong here—after all, mathematics is *episteme*, and its subjects are real.

Aquinas continues:

> At other times, what the name signifies is not a likeness of a thing existing outside the soul, rather, it is something which follows upon the way [the intellect] understands a thing outside the soul: and the intentions which our intellect discovers are of this sort (e.g., what the name *genus* signifies is not a likeness of some thing existing outside the soul, rather, because the intellect understands *animal* as being divided into many species, it attributes the intention of *genus* to it). Now, although the proximate foundation of this sort of intention is not in the thing but in the intellect, still, its remote foundation is the thing itself. And so the intellect which discovers these intentions is not false. And so it is with all other things which follow upon our mode of understanding, such as the abstraction of mathematicals, and the like.[3]

[2] *Scriptum Super Sententiarum*, I 2 3. "Aliquando enim hoc quod intellectus concipit, est similitudo rei existentis extra animam, sicut hoc quod concipitur de hoc nomine homo; et talis conceptio intellectus habet fundamentum in re immediate, inquantum res ipsa, ex sua conformitate ad intellectum, facit quod intellectus sit verus, et quod nomen significans illum intellectum, proprie de re dicatur."

[3] Ibid., "Aliquando autem hoc quod significat nomen non est similitudo rei existentis extra animam, sed est aliquid quod consequitur ex modo intelligendi rem quae est extra animam: et hujusmodi sunt intentiones quas intellectus noster adinvenit; sicut significatum hujus nominis genus non est similitudo alicujus rei extra animam existentis; sed ex hoc quod intellectus intelligit animal ut in pluribus speciebus, attribuit ei intentionem generis; et hujusmodi intentionis licet proximum fundamentum non sit in re sed in intellectu, tamen remotum fundamentum est res ipsa. Unde intellectus non est falsus, qui has intentiones adinvenit. Et simile est de omnibus aliis qui consequuntur ex modo intelligendi, sicut est abstractio mathematicorum et hujusmodi."

Our intellectual mode of understanding gives rise to this account of logical—second—intentions. Here, though we obviously conceptualize *genus* and *species* and other purely logical notions themselves, we do not regard them as being conformed to external reality, unless *mediately*. As Aquinas says, "[this intention's] remote foundation is the thing itself." This difference explains why the sophistical argument *animal is a genus, Socrates is an animal, therefore Socrates is a genus* is invalid, as one shifts from *first* to *second* intentions in the reasoning. This remote connection to reality, however, is sufficient for the truth of purely logical claims (*animal is a genus*).

Importantly, however, Aquinas explicitly includes the 'abstraction of mathematicals' among second intentions—which is to say that they have no proximate foundation (*proximum fundamentum*) in reality, only a remote one. This is to say, though mathematical claims and conclusions are true, they are not true of *things* so much as of the way *we* necessarily *think about* things. There are no genera and species *in the world*. And even though there is a real relation between matter and form—which relation founds the distinction *we understand* between genus, difference, and species[4]—logical science (*rationalis scientia*) is not about these things in the same way as physical science is about its objects. Cows and spiders would still be, and be what they are, were there no human minds to know them. Not so genus, species, and difference.

There remains one more way in which our concepts can be related to the external world:

> At still other times, what a word signifies has neither a proximate nor a remote foundation in a thing (for example the concept of a Chimera, since it is neither a likeness of a thing existing outside the soul, nor does it follow upon [the intellect's] way of understanding a natural thing): and therefore this conception is false.[5]

As it turns out, each of these three ways has its mathematical proponents. Plato and Aristotle, for example, explicitly defend a mathematics of

[4] We have seen this real difference in Aristotle, too. Recall our discussion of parts of a form and matter in mathematical definitions, above.

[5] Ibid. "Aliquando vero id quod significatur per nomen, non habet fundamentum in re, neque proximum neque remotum, sicut conceptio Chimerae: quia neque est similitudo alicujus rei extra animam, neque consequitur ex modo intelligendi rem aliquam naturae: et ideo ista conceptio est falsa."

the first sort, while Aquinas (apparently) sets the mathematical sciences alongside logic. Finally, though this remains to be investigated, there are those who regard the mathematicals and claims made about them as *useful fictions*.[6]

If Aquinas sees the mathematical sciences as being about abstractions directly conforming to physical substances, why does he group mathematical conceptions with logical intentions? If a geometer considers the physical circle, though *not as* physical, why insist that the concept, circle, arises only *remotely* from such substances, is *immediately* due to the way we think about quantity?

Near the end of the last chapter, we saw in Aristotle a resolution to the so-called *precision-problem* facing (especially) geometry. Given constructive demonstrations, we can discover what is true—albeit materially—in the realms of continuous and discrete quantity. It need not be about the *physical* circle that we prove that *a triangle in a semicircle is right*, rather, we *construct* the circle, *inscribe* a triangle in its half, and then *demonstrate* that it has a right angle. If physical mathematicals in general do *not* exhibit the properties demonstrated of them, it need not be because the properties are not *there* (in the same manner in which we say genus and species are not *there*). Real and continuous quantity in one, two, or three dimensions *admits of* the mathematical line, straight line, and circle in a way in which physical realities will never *admit of* genus, species, and the like, even granting that the mathematical line, straight line, and circle come to be and to be understood only by way of construction, in which (as Aristotle says) the intellects *acts*.

We will have more to say on this topic soon, but it seems to accord with both the received Thomistic accounts of the mathematicals and the passage just cited that *some* of what we term *mathematicals* fall into, and are understood by way of, the first sort of conceptualization mentioned (Aquinas is surely not saying that continuous quantity in one, two, and three dimensions and numerical plurality are second intentions having no proximate foundation in reality), while others come about and are understood by way of the constructive acts of an inquiring mind.

An objection arises, that one might justify extensions to the natural sciences in the same way (for physical matter is also in potency to being many things—even things which are not and have never been. Why not allow for mental constructions of living and nonliving substances and subsequent

[6] Notably, Hartry Field.

investigations of their properties? Such constructions, though having no immediate foundation in reality, would still have a remote foundation in the potentiality of matter, and hence meet the same conditions for being *sciences* in the proper sense as do the mathematical sciences.

While the complete answer to these questions will come later, we can say two things at least. First, *mental construction* and *physical substance* do not seem to accord. If a physical substance is to be realized, it would require a *physical* realization, not a mental one. The same does not appear to be the case for constructables which have already been abstracted from their physical conditions. There, construction does seem possible. Secondly, the potency of matter for being or becoming substance is actualized differently from the potentiality inherent in quantity, as such. As to the *coming-into-being* of forms—whether substantial of accidental—Aristotle says:

> Our argument shows that the form of a substance does not come into being—but not only [substance]. The same argument shows this about all the primary [things], that is, quantity, quality, and the other categories. For just as the bronze sphere is generated but not the sphere nor the bronze, (and so, too, for bronze if it were generated—*its* form and matter must always pre-exist), so it is with the *what* and the *quality* and the *quantity* and the other categories. For it is not the quality that is generated but wood of that quality, not the quantity but that quantity of wood or an animal of that size. And we can hereby learn a property of substance: that there must *actually* pre-exist some other substance which brings it into being—for example, [another] animal if an animal is being generated—but a quality or quantity need pre-exist only *potentially*.[7]

While we will soon take up the question whether mathematical constructability is naturally limited or not, it is clear that Aquinas, like Aristotle, sees no incompatibility between the necessity of constructive

[7] *Metaphysica* 1034b7–19. οὐ μόνον δὲ περὶ τῆς οὐσίας ὁ λόγος δηλοῖ τὸ μὴ γίγνεσθαι τὸ εἶδος, ἀλλὰ περὶ πάντων ὁμοίως τῶν πρώτων κοινὸς ὁ λόγος, οἷον ποσοῦ ποιοῦ καὶ τῶν ἄλλων κατηγοριῶν. Γίγνεται γὰρ ὥσπερ ἡ χαλκῆ σφαῖρα ἀλλ᾽ οὐ σφαῖρα οὐδὲ χαλκός, καὶ ἐπὶ χαλκοῦ, εἰ γίγνεται (ἀεὶ γὰρ δεῖ προϋπάρχειν τὴν ὕλην καὶ τὸ εἶδος), οὕτως καὶ ἐπὶ τοῦ τί ἐστι καὶ ἐπὶ τοῦ ποιοῦ καὶ ποσοῦ καὶ τῶν ἄλλων ὁμοίως κατηγοριῶν: οὐ γὰρ γίγνεται τὸ ποιὸν ἀλλὰ τὸ ποιὸν ξύλον, οὐδὲ τὸ ποσὸν ἀλλὰ τὸ ποσὸν ξύλον ἢ ζῷον. Ἀλλ᾽ ἴδιον τῆς οὐσίας ἐκ τούτων λαβεῖν ἔστιν ὅτι ἀναγκαῖον προϋπάρχειν ἑτέραν οὐσίαν ἐντελεχείᾳ οὖσαν ἢ ποιεῖ, οἷον ζῷον εἰ γίγνεται ζῷον: ποιὸν δ᾽ ἢ ποσὸν οὐκ ἀνάγκη ἀλλ᾽ ἢ δυνάμει μόνον. Emphases mine.

demonstrations in mathematics and meeting the requirements set out for *episteme*, in the strict sense. For geometrical demonstrations make frequent use of them:

> Sometimes the reason *why* finally comes down to what something is, that is, to the definition, as is clear with all immobile things—for example, the mathematicals. Here, the reason *why* comes down to the definition of the straight or the commensurate or some other thing which is demonstrated in mathematics. For since the definition of a right angle is *what is formed by one line falling upon another such that it makes the angles on both sides equal, if* one were to ask why this angle is right the answer would be because *it is formed by one line making the two angles on each sides equal.* And so it is in other cases.[8]

In Aquinas' example, the definition of the right angle is what he would call *quasi operative*. *Note the active terms:* the angle is *formed* by one line falling on another and thereby *making* an angle. This example is very much like the passage we cited above, in which Aquinas introduced us to the *operative demonstrations* employed by mathematicians in their sciences.[9]

The necessity of constructability among the mathematicals serves to mitigate our initial confusion regarding the above passage from Aquinas' *Commentary on the Sentences of Peter Lombard*. Aquinas is not, thereby, not a realist (or even partially an anti-realist), whether (1) because the overwhelming textual evidence is in favor of the traditional account of the mathematical sciences as standing alongside the natural sciences and metaphysics,[10] or (2) because even his logic is of a realist sort, or (3)

[8] *In Physicorum,* II 10239. "Quandoque enim propter quid reducitur ultimo in quod quid est, idest in definitionem, ut patet in omnibus immobilibus, sicut sunt mathematica; in quibus propter quid reducitur ad definitionem recti vel commensurati vel alicuius alterius quod demonstratur in mathematicis. Cum enim definitio recti anguli sit, quod constituatur ex linea super aliam cadente, quae ex utraque parte faciat duos angulos aequales; si quaeratur propter quid iste angulus sit rectus, respondetur quia constituitur ex linea faciente duos angulos aequales ex utraque parte; et ita est in aliis. Emphases mine.

[9] *In Posteriorum Analyticorum*, I 2 17, where one finds the example *on a given line to construct an equilateral triangle* and Aquinas' explanation of how mathematicals constructively demonstrated to exist can serve as subjects of their own (demonstrated) properties.

[10] Aquinas famously devotes two separate questions in his *Commentary on the 'De Trinitate' of Boethius* to this division of the sciences and to their methods.

because a complete reading requires that he ascribe to the intellect *some* role in establishing *that* the mathematicals—some, at least—do, in fact, exist.

Mathematical Ideals

We have already seen in Aristotle and Aquinas a tendency to *idealize* the geometricals. After all, unlike the mathematical line, the physical one does not touch a circle at a point—to say nothing of whether physical lines are straight or round-ish figures can be circular in the first place. On the other hand, the precision-problem seems less of a problem when we consider the abstractions and constructive demonstrations constituting arithmetic.

In his *Commentary on Aristotle's Physics*, Aquinas mentions that "*number* signifies that there are many things in one."[11] Of course, we *know* that number is a *many*. Moreover (as we saw with Aristotle, above) the number—any mathematical, really—need also *be* one.[12] There is yet another sense of *one* operative here beyond the unity of number itself, which is, of course, the *unit*. Aristotle says, in defining number, that "*Plurality* is a sort of genus of number: for number is *a plurality measured by the unit*."[13]

If we were to seek a starting-point for arithmetic as traditional philosophy presents it, we need look no further than the numerical unit which, though not itself a number,[14] is the measure of all numbers. Surprisingly, though, Aristotle will not allow that some ordinary numbers (like one,

[11] *In Physicorum*, III 12 392. "Numerus enim hoc significat, quod sint aliqua plura uno."

[12] This is the sense of the *in one* in the above quote.

[13] *Metaphysica*, X 7 1057a2–4. τὸ δὲ πλῆθος οἷον γένος ἐστὶ τοῦ ἀριθμοῦ: ἔστι γὰρ ἀριθμὸς πλῆθος ἐνὶ μετρητόν. See also *In Metaphysicorum*, X 8 2080. The emphases are mine. Aquinas identifies the unit as the principle of number in many places, especially when he distinguishes it from that sense of one which is convertible with being, adding to undivided being the ratio of being a measure.

[14] Aristotle and Aquinas deny that the unit is a number. See *Metaphysica*, X 7 1056b3–33, and *In Metaphysicorum*. Aristotle calls the number two the *first plurality* at 1056b26. At *Analytica Posteriora*, II 13 96a35–6 (cf. also Aquinas' *In Analyticorum Posteriorum*) Aristotle describes a prime in two ways: as neither *measured by* a number nor *composed of* numbers. By this account, then, the unit cannot be a number, since it both measures and composes primes. A principle (such as the unit is) can, but need not, be the same kind of thing of which it is a principle. Matter, a principle of substance and of change, is neither substance nor change. On the other hand, the revolution of the moon—a principle of the tides—is itself a change in place.

which nearly everyone would call a number) are numbers at all, using the notions of *measure* and *composition* to support his conclusion. By what criterion can sound construction demonstrations be distinguished from unsound ones, if we are to begin constructing numbers, as we must?

Mathematical Freedom

As we mentioned above, the late-eighteenth- and early-nineteenth-century philosopher and mathematician, Georg Cantor, was decidedly a realist of a platonic sort.[15] As such, we would have expected him to be—and he was—very much concerned with establishing the *reality* of those mathematical objects he thought to be legitimate and to distinguish them from those he regarded as illegitimate. He describes that process in this way:

> The procedure for the correct construction of concepts is in my opinion everywhere the same; one sets up a thing [*Ding*] having no properties, which at first is nothing more than a name or sign A and gives this *according to some law*, different and even an infinite number of properties, whose meaning is known through ideas already existent which may not contradict one another.[16]

This test for mathematical legitimacy sounds reasonable. Applying it to arithmetic, in order to justify numerical demonstrative constructions on the basis of these restrictions, we not only have to determine whether such constructions are self-consistent, but also whether they are consistent with mathematical notions already in place.

Adhering to the account just provided from Aristotle and Aquinas, the first of all arithmetic notions is the unit, for it is found in the very definition of number. If two is, as Aristotle insists, the *first plurality*—and we grant that any natural number can be generated from, and consistently with, the unit—what might be said in favor of numerical extensions to the natural numbers in the light of our first principle? For example, can we legitimize zero?

Zero is easily constructed by means of an established operation (*law?*) with which we are all familiar—subtraction. Apparently, subtracting one

[15] Georg Cantor exemplifies mathematicians who seek to justify extensions to the natural numbers. We will return to Cantor in Part III.

[16] Georg Cantor, *On Infinite,Linear Point-Manifolds*, trans. George Bingley (Annapolis: Classics of the St. John's Program, 1942), 148, note 7 of § 8. Emphasis mine.

number from another yields a number—at least the result *differs from* the minuend by the subtrahend, in fact, by definition, the result *is* the difference.[17] If things which differ are the same in kind, the result of subtracting one number from another is a number. The conclusion seems to be that zero and the negative integers are legitimate numbers, with zero being the number we construct when we subtract a number from itself, and the negatives, in turn, numbers we construct when we subtract a greater number from a lesser.

In a like manner, we may take one number and measure another by means of the first[18]—this is the essence of division. Aristotle, Euclid, Aquinas—we *all* call numbers which are measured by the unit alone *prime*. The number seventeen is prime, because no number measures *it* a number of times.[19] Yet why need division be limited to numbers which measure only a number of times?[20] Absent this limitation, we can legitimize (properly) fractional numbers. For example, though two measures four two times it measures five *differently*, that is, *two and one-half* times. While two is like other Aristotelian and Thomistic numbers which are measured by the unit, the newly constructed one-half appears to be like zero and the negative numbers mentioned above. It is what remains following the operation of dividing a lesser whole into greater parts. Said in this way, of course, one-half sounds impossible—*is* impossible, at least *physically*.

Notice that these constructive demonstrations yielded results only once we set aside common-sense limitations on these basic arithmetical operations. Why would we ever *think* that more could be subtracted from less or that a lesser whole could be divided into greater parts? It is by the *unlimited* application of these operations that we are able to construct zero, negative numbers, and fractional numbers. But why not allow ourselves such freedom? After all, the benefits of even these few extensions to the natural numbers are obvious. Moreover, this freedom appears to have

[17] Put another way, we recognize that the counting numbers are not closed under subtraction, but the integers are.

[18] For both Aristotle and Aquinas, though all numbers are (by definition) measured by the unit, numbers can also measure other numbers. Cf. Aristotle's definitions of *prime* in *Analytica Posteriora*, II 13.

[19] Euclid also distinguishes three types of non-prime—that is, composite—numbers, all of which are measured by other numbers. See Euclid, *The Elements,* VII.

[20] That is, without a remainder.

only a limited contradictory result.[21] Following Cantor's rule, then, we should declare division by zero as having an *illegitimate* result (it contradicts earlier notions) and declare zero, the integers, and the rationals, to be legitimate numbers.

Other basic arithmetical operations we might take as—using Cantor's term—*laws* according to which we legitimate our constructive demonstrations in arithmetic are addition, squaring, and taking the square root. Let us investigate what their unlimited application might yield.

No one denies that the sequence of natural numbers has *some* sort of infinity to it. For every natural number *n* one can always construct a greater, namely *n+1*. This infinity has been variously described—as actual or potential, for example. We saw very early on that Aristotle does not allow for an actual infinity among numbers. Aquinas concurs, and in many places.

On the other hand, if we were to regard the natural numbers as a sort of *open-ended* whole, constructively generated by adding *without* limits, a *further* addition can be conceived, resulting in the creation of a new number and a new kind of number—a *transfinite*.

> However contradictory it might be to speak of a greatest number of class (I) [the class of integers], there is nevertheless nothing offensive in thinking of a new number which we shall call ω, and which will be the expression for the idea that the entire assemblage (I) is given in its natural, orderly succession. (Just as ν is an expression for the idea that a certain finite number [*Anzahl*] of unities is united to form a whole). It is accordingly permissible to think of the newly created number ω as the *limit* to which numbers ν approach, if by it nothing else be understood than that ω is the *first* integer which succeeds all numbers ν, that is, it is to be regarded as greater than every one of the numbers ν.[22]

Of course, the idea *greatest integer* is a contradiction—as we said, for any number *n*, we can generate a greater, *n+1*, by way of addition. Still, Cantor thinks we *can* conceptualize *integers produced by addition*. Because the number of this collection cannot be any of its members (as we just saw) ω must be a new number and a new *kind* of number—it is *not* a finite

[21] Notably, however, division cannot be freely applied to these *new* numbers without contradiction, as is clearly seen in the instance of division by zero.

[22] Georg Cantor, *On Infinite, Linear Point-Manifolds*, 131–2.

integer. Other *transfinite numbers* and orders of such numbers are shown to exist along the same lines.

Freely *taking the square root* of natural numbers eventually leads to numbers like the square root of two—a so-called *irrational* number. Notice how we have extended our original concept of *number*. In this case, the number is incommensurable with the unit. Numbers which include the irrationals are said to be *real*. If we *take the square root* without limiting ourselves to the natural numbers, we can construct so-called *imaginary* numbers, such as *i*, the square root of negative one. Complex numbers are produced in a similar way.

In accordance with the *laws* of arithmetic, then, mathematical operations freely and recursively applied to the natural numbers legitimize calling such products *numbers*. And why not? Rather, why do Aristotle and Aquinas *disallow* them if they themselves insist upon the need for constructive demonstrations in mathematics? Even apart from Aquinas' account of the threefold relation of concepts to thing,[23] both philosophers hold that (at least some) mathematicals are not *discovered* but *constructed*. As for constructible mathematical objects in general, then, so for constructing these other numbers.

Moreover, our very conception of numbers is *linked to* their genesis. The *eighty* we name by reference to one of its factors is also called *quatre-vingt*—four twenties—in French. We naturally arrive at *two* by adding the unit to itself—the simplest possible operation. Finally, then, are not all numbers *made* anyway? Why can we not make them as we please?

While Aristotle distinguishes art from *episteme* and even constructive proofs from demonstrative one, Aquinas goes beyond this and accounts for *mathematical* arts as distinct from *mathematical* sciences.[24] Adding the unit to itself yields two, no doubt, but what two *is* contains no reference to how it came to be. Being a number—even if it happened to have been constructed—is not being the result of mathematical operations. The operations are *accidental* to the natures so produced. Following Aristotle's and Aquinas' account of the mathematicals, we would properly abstract from this in our consideration.

[23] This distinction already implies that the intellect is the immediate source of some of our concepts.

[24] See especially Thomas Aquinas, *In Librum Boetii De Trinitate Commentarium* (Rome: Marietti Editori Ltd., 1954), V 1. Hereafter *In De Trinitate*. The distinction itself goes back at least to Plato, who regarded the liberal arts—including the mathematical arts—as a proper part of the education of the guardians of the republic, as we saw above.

More broadly, Aristotle and Aquinas do not see artificial things (things made) as essential realities in the first place. Artificial things have no natures. They are merely identical with what composes them.[25] For example, we might name a collection of metal and wood a *desk* but it has no nature of its own. *Its* properties are the properties of the natural substances out of which it is made.[26]

As for *two*, it has a distinct nature and its own properties. It is not an artificial *thing*, product of addition though it may be. This is precisely what Aristotle and Aquinas mean in saying that "two is one." *Two* is not a collection or aggregate but has its own unity and properties which differ from its components, considered as such. In fact, the properties of the unit are *incompatible* with those of *two*.[27] To *conceptualize* two as *the unit added to itself*, then, is to falsify its nature and to regard it as an artificial thing.[28] On the other hand, we do *make* two by adding one to one. How can *two* have a nature and yet be a product of the arithmetical art?

On the other hand, not all products are artificial, for example, when we initiate the composition of hydrogen and oxygen using a spark.[29] Similarly, though we might *think of* two as 1 added to 1,[30] *being* the number two is not *being* one added to itself. Two, as such, remains one.

The art of arithmetic constructs numbers, whose properties we then investigate scientifically. Generally, mathematics constructs (at least some) mathematicals, whose properties we investigate scientifically. Now, we have seen that some constructions are mathematically illegitimate.[31]

[25] One clear *locus* for this distinction is *Physica*, II 1.

[26] Desks, as such, neither burn nor are they heavy, though their components might.

[27] It follows that the unit cannot be actually present within *two*, unlike the way in which wood is present in a desk.

[28] There is more. As Aquinas says: "Whence it is said in the fifth book of the *Metaphysics* that the substance of six consists in being six times one and not in being two times three or three times two: since it would then follow that there would be many definitions and many substances of one thing; since one number arises from different parts in different ways." *In Physicorum*, III 12 392. "Unde dicitur in V Metaphys. quod substantia senarii est in hoc quod sit sexies unum, non autem in hoc quod sit bis tria vel ter duo: quia sequeretur quod unius rei essent plures definitiones et plures substantiae; quia ex diversis partibus diversimode consurgit unus numerus."

[29] To argue that it is the spark, and not us, which makes the water is to miss the point: *all our makings require natural processes.* Harder substance naturally wear away at softer ones: yet this is the essence of sculpting.

[30] One might just as well think that being spark-produced is of the essence of water.

[31] For example, any number divided by zero.

Logically speaking, that (even artificially produced) natural numbers are legitimate does not imply that other such productions will also be. On the other hand, how can we legitimize any, if it is not in virtue of the laws of mathematics?

At the very end of his *Elements*, Euclid supplies a clear example of *internal* limits to constructive demonstrations. In Proposition Eighteen of Book Thirteen, he constructs the five regular solids and proves that only these five can be constructed.[32] Necessarily five—and only five—regular solids follow from the assumption of geometrical principles. This necessity is rooted in geometry's first principle, continuous quantity in three dimensions.

So, too, the construction of a number is rooted in the unit, the first principle of arithmetic, about which we must know *what it is* and *whether it is*.[33] In a manner analogous to Euclid's geometry, to scientifically investigate the properties of numbers we must first construct those numbers *necessitated* by the unit—*only these will be numbers in the proper sense*. It is not sufficient to construct by adding, subtracting, and so on. These are new creations, "whose meaning is known through ideas already existent which may not contradict one another."[34] Each new construction must be self-consistent, but also consistent with existing ones—*a fortiori* with the first principle of the science itself. This alone is *episteme*, as opposed to fabrication.[35]

Returning to geometry, we now see why the physical line and circle seem to have been transformed—idealized—into their geometrical counterparts. It is sufficient that geometry abstract *continuous quantity in three dimensions*[36] from physical substances to do its work: constructively demonstrating the existence of geometric figures and investigating their properties. Knowing *three-dimensionality* leads to knowing *surface* and *plane surface*, *line* and *straight line*, and *point*— the materials we need to

[32] Euclid, *The Elements*, XIII 18 and Euclid's remark immediately preceding XIII 18 lemma.

[33] Recall the conditions Aristotle lays out for *episteme*. The unit is his own example of the subject of *episteme*.

[34] This constitutes a return to Cantor's own procedure, introduced above.

[35] The analogy to the five regular solids is again *apropos*. Non-Euclidean geometries might construct other regular solids, but they do not begin with the three-dimensional continuum. *That* beginning entails *these* consequences.

[36] We might expand this subject-name to *continuous quantity in one, two, and three dimensions* were it not for the trivial analysis of three dimensions in two and one.

construct the geometricals—by analysis.[37] It turns out that idealization, a presumed transition from physically imperfect to perfect straight lines and circles, is really constructive demonstration. Only those geometrical lines, surfaces, and solids which validly *follow from* our notion of continuous quantity in three dimensions—which is really *in things* as we conceive it—are legitimate subjects of *episteme* in the geometrical order.

Mathematical Legitimacy

Cantor's discovery of the transfinite numbers had an impact upon the mathematicians of his day not only in the obvious way, that is, it affected what new directions their work as mathematicians might take, but also *philosophically*. Cantor himself was as concerned with justifying these numbers as mathematically legitimate as with addressing, and resolving, philosophical objections to the very *idea* of actual mathematical infinities, especially those he found in the writings of Aristotle.[38] Reactions from his contemporaries ranged from Leopold Kronecker, "God made the integers; all else is the work of man,"[39] to L.E.J. Brouwer, whose intuitionism was *predicated upon* a rejection of Cantor, to David Hilbert, who famously declared: "No one shall drive us out of the paradise which Cantor has created for us!"[40] Gottlob Frege himself noted that the issue would result in a damaging conflict within mathematics itself:

> Here is the reef on which it [mathematics] will founder. For the infinite will eventually refuse to be excluded from arithmetic, and yet it is irreconcilable with that [finitist] epistemological direction. Here, it seems, is the battlefield where a great decision will be made.[41]

Mathematics seems to evolve in the same way as human knowledge generally. More conservative views hold sway until later thinkers arise to

[37] If three then two and if two then one, uni-dimensionality *being* the straight line, whose limit is a point. So, too, with the others.

[38] Cantor devotes an entire section of *On Infinite, Linear Point-Manifolds* to refuting Aristotle.

[39] Quoted in Heinrich Weber, "Leopold Kroenecker," *Mathematische Annalen* 43 (1893): 19.

[40] David Hilbert "On the Infinite," in Benacerraf and Putnam, *Philosophy of Mathematics: Selected Readings*, 191.

[41] Quoted in Joseph Dauben, *Georg Cantor: His Mathematics and Philosophy of the Infinite* (Princeton, New Jersey: Princeton University Press. 1979), 225.

challenge the rigid notions of the day. Historically, this trend has been instanced in number theory, with natural numbers successively followed by zero and the integers, the rationals, the transcendentals and reals, and infinite and complex numbers.[42] Though a clear instance, as we said above, Cantor is not unique among mathematicians separately arguing for the *legitimacy* of their discoveries.

To Cantor, mathematics consists in developing methods for constructing its objects and discovering the laws governing their relationships. As to his metaphysics, the mathematicals[43] have an *intrasubjective* (immanent) reality and a *transsubjective* (transient) reality.

> Firstly, integers may be considered real in so far as they occupy an entirely definite place in our understanding on the basis of definitions, can be precisely differentiated from all other parts of our thought and stand in determinate relationships to those parts, and accordingly modify the substance of our thought (*Geistes*) in a determinate fashion … Secondly, reality can be ascribed to numbers in so far as they must be taken as an expression or image of the events of that outer world which is exterior to the intellect, as, for instance, the various number classes (I), (II), (III), etc., are representatives of powers which are actually found in corporeal and intellectual (*geistegen*) nature.[44]

It is no surprise, given what we said of him in the first chapters of this book, that the external (transient) reality of the mathematicals is a matter of concern for Cantor. To each legitimate mathematical conception there corresponds something objectively real:

> …these two spheres of reality are always found together in the sense that a concept said to exist in the first sense also possesses in certain and even in an infinity of ways a transient reality…[45]

[42] There is evidence for a broader scope even in ancient times, but that is not to-the-point here.

[43] This distinction applies to the reality of *all* conceptualizations. Georg Cantor, *On Infinite, Linear Point-Manifolds*, Part 5 § 8.

[44] Ibid., 114.

[45] Ibid. Discovering these transient correlates, he says, is "one of the most troublesome and profound problems of metaphysics" which "must frequently be left to times in which the natural development of one of the other sciences eventually reveals the transient meaning of the concept in question." See Ibid., § 8 note 7. This observation is an essentially platonic one. See note 6.

Even so mathematics is properly concerned with the *intrasubjective* reality of the mathematicals:

> The connection of both realities has its peculiar foundation in the *unity* (*Einheit*) of the *All* (*Alls*), *to which we ourselves belong*. A mention of this connection has only the value here of enabling one to derive from it a result which seems of very great consequence for mathematics, namely, that it (mathematics) in the construction of its ideas has *only* and *solely* to take into account the *immanent* reality of its concepts and has *no* obligation whatever to make tests for their *transient* reality.[46]

Though notable among mathematicians for his desire to also philosophize, Cantor is quick to distinguish the proper work of each. While the latter is all about the objective realities to which mathematical discoveries correspond, the former has no concern for these. Mathematicians need not wait upon philosophers for progress. This is especially so because transient correlates might *never* be discovered. In short, the mathematician is *free*:

> On account of this singular position which differentiates it from all of the other sciences and affords an explanation for the comparatively easy and unrestrained manner with which one may operate with it, it particularly deserves the name of *free mathematics* (*freien Mathematik*), a designation which, if I had my way, would take precedence over the usual expression 'pure' mathematics.[47]

On the other hand, he is not advocating mathematical license, either. As we have seen, mathematical progress must be self-consistent and consistent with what preceded it.[48] Still, he does (famously) declare that *external* restrictions contradict the very nature of mathematics:

> I regard every restriction on the urge to mathematical investigation as creating a much greater danger and all the greater since no justification can be advanced from the nature of the science; the nature (*Wesen*) of *mathematics* lies in its very *freedom*.[49]

Legitimization belongs to mathematics proper.

[46] Ibid., pp. 114–15.
[47] Ibid., p. 115.
[48] His own examples of mathematical progress are the rational, irrational, and complex numbers.
[49] Ibid., pp. 115–16.

There is another, secondary, legitimization to which Cantor does give some weight, however—its successful *application* to the physical world:

> There is no need, as I believe, of fearing any danger to the sciences in these principles, as many think: the conditions imposed under which the freedom of constructing numbers can be exercised are of such a kind that they permit very little room for arbitrary action (*Spielraum*); for every mathematical concept carries within itself the necessary corrective; if it is fruitless and purposeless, that is soon seen and it will because of little success be abandoned.[50]

Philosophy cannot set *a priori* limits to mathematical progress, though it may have a role in discovering the *transient* reality of concepts *already shown to be real*—that is, *intrasubjectively*.

Cantor acknowledges that some think such freedom is dangerous. *Finitists* consider legitimizations based on consistency as prone to *error*.[51] Perhaps (they say) such developments are *merely* intrasubjective—definite and distinct from other discoveries with which they are in accord. Cantor maintains both, insisting that progress *will correlate* with the objectively real and that the hard work of correlating them may be ultimately unsuccessful. Not a concern, though, since it is not a condition for mathematical legitimacy anyway.

Cantor believed that philosophy's intrusions into free mathematical inquiries find their origin in Aristotle and in subsequent scholastic developments of his thought.[52] We have seen this specifically in regard to the development of the transfinite numbers, but the same argument weighs in favor of the other developments he mentions. Most mathematicians (by far) do not regard the legitimacy of higher mathematics as being in question, if they think about the matter at all. Among those who do, a few have argued that, were Thomas Aquinas alive today, he—and possibly even Aristotle—might very well have embraced it.[53]

[50] Ibid., p. 115. Cantor has the theory of functions of Gauss, Cauchy, and others, in mind, saying: "…which [theory], although created and erected *in full freedom without transient purpose* nevertheless reveals its transient meaning in applications to mechanics, astronomy, and mathematical physics, as was to be expected." Ibid., p. 116.

[51] Ibid., § 4. It is on the reef of mathematical truth that this particular controversy founders.

[52] Cantor mentions the scholastics only briefly in *On Infinite, Linear Point-Manifolds*, but began to focus on their arguments against the actual infinite in the years which followed.

[53] Thomas Anderson, "Aristotle and Aquinas on the Freedom of the Mathematician," *The Thomist* 36 (1972): 231–55, Armand Maurer, "Thomists and Thomas Aquinas on the Foundation of Mathematics," and Adam Drozdek, "Number and Infinity: Thomas and Cantor," *International Philosophical Quarterly* 39 (1999): 35–46, are among these. We will consider these thinkers more directly in Chap. 13.

Do such unwarranted restrictions necessarily follow from Thomistic principles, or are modern mathematical discoveries consistent with them? It is our contention that, given the nature of the unit and continuous quantity in three dimensions—the ultimate source of mathematical truths—many extensions to simple geometry and arithmetic are not sciences, *episteme*—in the traditional sense. What *epistemic* status they do have we will address later.

What exactly is it about traditional accounts that excludes such things? As we have already seen:

> The mathematician makes a study of things which result from abstraction, for he studies things by taking [from them] everything sensible, such as heaviness and lightness… and he leaves only quantity and the continuous (according to one, two, or three dimensions) and the attributes of these as quantity and as continuous.[54]

Unlike Cantor,[55] consistency with *something objective* is an a priori condition for *episteme—scientia*. *External constraints* upon free mathematical inquiry are necessary to the kind of knowing *episteme* is. Mathematics must be internally consistent, of course, but beyond this it must maintain a connection to *the* real.[56] Cantor believes there exist point-for-point correlations between mathematical discoveries and transient reality.[57] Were Aquinas to insist upon *such* a criterion for legitimacy, mathematicians would need to point physical correlates to each new arithmetic and geometrical construction—not possible even for the most basic of geometrical constructions, at least. But that is not the criterion for *episteme*. Aquinas—and Aristotle—take a middle position here, requiring only that constructive

[54] *Metaphysica*, XI 3 1061a29–35. ὁ μαθηματικὸς περὶ τὰ ἐξ ἀφαιρέσεως τὴν θεωρίαν ποιεῖται (περιελὼν γὰρ πάντα τὰ αἰσθητὰ θεωρεῖ, οἷον βάρος καὶ κουφότητα… μόνον δὲ καταλείπει τὸ ποσὸν καὶ συνεχές, τῶν μὲν ἐφ' ἓν τῶν δ ἐπὶ δύο τῶν δ ἐπὶ τρία, καὶ τὰ πάθη τὰ τούτων ᾗ ποσά ἐστι καὶ συνεχῆ…)

[55] Recall that, for Cantor, the discovery of transient physical correlates to consistent mathematical constructions is an activity wholly other than, and independent from, mathematics.

[56] As it turns out, what Aristotle and Aquinas expect from mathematics is far less than the correlation-process Cantor describes above.

[57] Having real-world applications seems to be at least along-the-way of discovering such correlates, it seems.

demonstrations correlate with external reality *in principle*.[58] How would Aquinas work this out, specifically? Let us begin with arithmetic.

If numbers beyond the naturals entail dividing the unit or compromising its nature as a principle or measure,[59] whatever results we may achieve through their use and application, these constructs are not really numbers at all, as being inconsistent with the (admittedly real) fundamental principle of arithmetic.

Consider what it would *mean* for a number to be a *proper fraction*. While symbols are inherently useful, it sometimes happens that they can lead us astray. Take the superscripted number two, conventionally indicative of *squaring*. 2^2 is two multiplied by two, thus ^{2} symbolizes a fundamental arithmetic *operation*, whereas 2 symbolizes a number—an *operand*. We also symbolize the *results* of mathematical operations—things like the resultant number two itself, whose symbol (again) is 2. In such simple examples it is quite easy to keep separate what stands for an *operation* and what stands for the *number* upon which we are operating: something we obviously wish to do, for it makes no sense to square squaring[60] or to *two-ify* the number two.[61]

We also sometimes combine symbols to represent a number. Squaring the square of the number two is helpfully expressed as $(2^2)^2$. No one intends thereby to square the operation of squaring, rather, we only wish to square a resultant number—it is only because what is represented within the parentheses *is* a number that we can square it at all.

Similarly, the operational symbol / is a convention for measuring one number by another, the number being measured preceding the number measuring it.[62] 20 / 4 stands, thereby, for the measuring *of* the number

[58] This will become clearer when we take up their notion of the potentially—or virtually—real. To be fair, exactly those things Cantor feared would impede legitimate mathematical freedom were *embraced* as reasonable constraints by some of his own contemporaries. In fact, the very *development* of rival schools of thought in the philosophy of mathematics during the late-nineteenth and early-twentieth centuries was due to controversies surrounding this very issue. See Benacerraf and Putnam, *Philosophy of Mathematics*. Where *intuitionism* insists upon finite constructability, the more liberally minded *logicists* and *formalists* are content with indirect proofs. The dispute over mathematical legitimacy and freedom of inquiry is also a thoroughly modern affair.

[59] Recall Aristotle's and Aquinas' definition of the numerical unit, above.

[60] As if we were to consider 2 as both an operation and an object upon which to perform that operation.

[61] As if we were to consider 2 as both an object and an operation to perform upon it.

[62] Division is as commonly represented in other ways, but this is not relevant here.

twenty *by* four. Here, too, we do not hesitate to apply subsequent operations to the result number, for example, $(20 / 4)^2$. Still, to get to the heart of the matter, this application has serious limitations.

No one *operates on an operation*. As we said, it makes no sense to square squaring (or to divide division or add addition, for that matter). The most we can do is to operate upon resultant numbers. Yet, given the ease with which we symbolize both numbers and operations, it may happen that a *supposed* resultant is an *incomplete operation*. No one would hesitate to symbolize squaring our measuring of the number one by the number three by $(1 / 3)^2$. We might even think it justified to symbolize the result of the first operation by ζ[63] and to express the resultant number by the symbols ζ^2. The problem is that ζ symbolizes, not a number, but an *incomplete* operation—dividing (*attempting* to divide, really) the arithmetical unit into three parts yields a value with an infinite decimal expansion.[64] It is possible, then, to inadvertently symbolize operations as numbers.

Moreover, ζ is an artifact, a properly human construct, and has no *nature*. In light of seeing that $1 + 1$ is not the same as the number 2, we can see each number has a nature distinct from that of every other. ζ, on the other hand, is inherently incomplete.

There is no need to think that ζ will necessarily lead to contradictory results or be practically unsuccessful. On the other hand, neither is a sufficient condition for concluding that ζ is a number, anyway. It follows that $\zeta + 1 = 1 + \zeta$ is not demonstrable within the *science* of arithmetic.[65] As we shall soon see, ζ is demonstrably inconsistent with the arithmetical unit as well.

Mathematical Illegitimacy

David Hilbert, a proponent of formalism, argues against this, saying that *incomplete processes* (concepts only intelligible in terms of an infinite number of steps, for example) can be justified by (1) *formalizing* statements made about them and (2) reducing these *ideal statements* to finite processes. By his account, it makes no sense *not* to allow such concepts into mathematics:

[63] We chose ζ arbitrarily.

[64] The common form of such a value signifies its incompleteness, as, operationally, it is infinitely recursive. Digital calculators deal with this by setting an arbitrary limit to certain operations. The term *anytime algorithm* is used to describe operations which result in successively more precise approximations.

[65] Exactly what character such a claim has will be addressed in the next section of this chapter.

> The following [objection] recently appeared: Although it may be possible to introduce a concept [presumably, into one's system] without risk, i.e., without getting contradictions, and even though one can prove that its introduction causes no contradictions to arise, still the introduction of the concept is not thereby justified. Is not this exactly the same objection which was once brought against complex-imaginary numbers when it was said: "True, their use doesn't lead to contradictions. Nevertheless their introduction is unwarranted, for imaginary magnitudes do not exist"? If, apart from proving consistency, the question of the justification of a measure is to have any meaning, it can consist only in ascertaining whether the measure is accompanied by commensurate success. Such success is in fact essential, for in mathematics as elsewhere success is the supreme court to whose decisions everyone submits.[66]

We simply *cannot* abandon those parts of mathematics which have had such tremendous calculative power.[67] If success (second only to consistency) becomes a criterion for mathematical legitimacy, Hilbert's program, or something like it, becomes necessary. It again comes down to whether one's mathematics is also a realism.

Is ζ compatible with the arithmetical unit, then? Obviously no—properly fractional numbers divide the unit. Thus, Aquinas:

> Between two points there happens to be a middle; for every line is a middle between two points: but between two unities there need not be a middle. For it is clear that between the two unities which constitute duality and the first unity itself there is no middle.[68]

So, too, from Book Three of his *Commentary on Aristotle's Physics*:

> Every unit, insofar as it is one, is indivisible, for example, an indivisible man is one man and not many. It is necessary to resolve every number into the

[66] Hilbert, *On the Infinite*, 184. Tool-makers are also *legitimate* to the extent that they successfully manufacture tools. Even so, it does not follow that mathematics is an art. *Episteme* also has success in its applications.

[67] The infinitesimal calculus is a prime example. Cf. Ibid., 187–8. Cantor differs from Hilbert here only in insisting that legitimate mathematical objects have transient correlates.

[68] Thomas Aquinas 1965 *In physicorum*, V 5694. "Inter duo puncta contingit esse aliquid medium; omnis enim linea est media inter duo puncta: sed inter duas unitates non necesse est esse aliquod medium. Patet enim quod inter duas unitates, quae constituunt dualitatem, et primam unitatem, nihil est medium." Cf. *Physica*, V 3 227a29–32. "There can be something between [points] (for every line is between points,) but it is not necessary [that there should be something between units]: for there is nothing between two and one." καὶ τῶν μὲν ἐνδέχεται εἶναί τι μεταξύ (πᾶσα γὰρ γραμμὴ μεταξὺ στιγμῶν), τῶν δ' οὐκ ἀνάγκη· οὐδὲ γὰρ μεταξὺ δυάδος καὶ μονάδος.

> unit: which is clear from the very notion of number. For 'number' signifies that there are many things in one: yet all pluralities, surpassing one more or less, are determinate species of number. Whence, since unity is of the notion of number, and indivisibility is of the notion of unity, it follows that the division of number must cease at the indivisible term.[69]

Nothing prevents our dividing three into three ones, whereas the opposite attempts the impossible, dividing the indivisible. 1 / 3, and proper fractional numbers generally, are incompatible with arithmetic's first principle and cannot be validly demonstrated by construction.[70]

A similar argument applies to zero and the negatives. Subtracting *x* from *y* entails resolving *y* into its units and counting up how many units remain when *x* units are removed. If no units remain (zero) or fewer than none (negatives) there *is* no unit, and so no principle, and so no number having that principle. Moreover, number is a *plurality* measured by the unit, such that even the unit—not being a many—is not a number. *A fortiori*, zero and the negative numbers are not numbers in the proper sense.[71]

As for the irrationals Aristotle is explicit: "Number is commensurate. *Number* is not said of what is not commensurate."[72] Aquinas elaborates:

> It happens that some *continuous* quantities have a proportion to each other, yet not according to a number, whether determinate or indeterminate. For

[69] Ibid., III 12392. "Omne unum, inquantum unum, est indivisibile, sicut homo indivisibilis est unus homo et non multi. Quemlibet autem numerum oportet resolvere in unum: quod patet ex ipsa ratione numeri. Numerus enim hoc significat, quod sint aliqua plura uno: quaelibet autem plura excedentia unum plus vel minus, sunt determinatae species numerorum. Unde cum unum sit de ratione numeri, et de ratione unius sit indivisibilitas, sequitur quod division numeri stet in termino indivisibili." Cf. *Physica*, III 7 207b6–8. "The unit is indivisible, whatever it may be (for example, a man is one man and not many,) but number is many ones and a certain quantity. Thus it is necessary to stop at the indivisible." αἴτιον δ' ὅτι τὸ ἕν ἐστιν ἀδιαίρετον, ὅ τι περ ἂν ἓν ᾖ (οἷον ἄνθρωπος εἷς ἄνθρωπος καὶ οὐ πολλοί), ὁ δ' ἀριθμός ἐστιν ἕνα πλείω καὶ πόσ' ἄττα, ὥστ' ἀνάγκη στῆναι ἐπὶ τὸ ἀδιαίρετον.

[70] 1 / 3 *can* designate a relation: the unit is to three as a third is to its triple. (For more on numerical relations, see Aquinas' masterful treatment of them, *In Metaphysicorum*, V 17 following *Metaphysica*, V 15.) Even so, such relations are not numbers, rather, they *presuppose* numbers.

[71] As we saw with fractional numbers, negative numbers, though demonstrably incompatible with unit, do have numerical functions (such as *to the left of* on a number line), but, again, such relations are not themselves numbers.

[72] *Metaphysica*, V 15 1021a5. ὁ γὰρ ἀριθμὸς σύμμετρος, κατὰ μὴ συμμέτρου δὲ ἀριθμὸς οὐ λέγεται.

> there is some proportion among all continuous quantities; yet it is not [necessarily] a numerical proportion. For of any two numbers there is one common measure, namely, the unit, which, taken many times, results in some number. But there cannot be found a common measure of all continuous quantities; rather, there are certain incommensurable continuous quantities: for example, the diameter of a square is incommensurable with its side. This happens because there is no proportion between it and its side like the proportion of a number to a number or of a number to the unit.[73]

Irrationals are therefore not measured by the unit, since, by definition, they cannot be expressed in terms of natural numbers—this is what it *means* to say they are incommensurable.[74] While this is sufficient to exclude them from consideration as numbers, irrationals also entail a division of the unit. Of what cannot be completely measured by the unit there must remain some portion less than the unit. The point is that *quantitative* relations need not be *numerical* relations, as Aquinas argues:

> When one quantity is said to be greater than another or is related to another as what contains is to what is contained, not only is this ratio not considered according to some definite species of number, it is not considered according to number at all, because every number is commensurable with every other. For all numbers have one common measure, which is the unit. But *contains* and *is contained* are not said according to a *numerical* measure. For, in relation to what is contained, the container is said to have so much and something more, and this, whether commensurable or incommensurable, is indeterminate. For whatever quantity be taken, it is either equal or unequal

[73] *In Metaphysicorum,* V 17 1020. Emphasis mine. "Contingit enim aliquas quantitates continuas habere proportionem adinvicem, sed non secundum aliquem numerum, nec determinatum, nec indeterminatum. Omnium enim quantitatum continuarum est aliqua proportio; non tamen est proportio numeralis. Quorumlibet enim duorum numerorum est una mensura communis, scilicet unitas, quae aliquoties sumpta, quemlibet numerum reddit. Non autem quarumlibet quantitatum continuarum invenitur esse una mensura communis; sed sunt quaedam quantitates continuae incommensurabiles: sicut diameter quadrati est incommensurabilis lateri. Et hoc ideo, quia non est proportio eius ad latus, sicut proportio numeri ad numerum, vel numeri ad unum." No length, howsoever small, can evenly measure both the side and the diagonal of a square, whose ratio is symbolized by √2, one of the best-known irrationals. Aristotle himself mentions a proof of this in *Analytica Priora* (Oxford: Oxford University Press, 1982) I 44 50a35–38, (hereafter *Analytica Priora*), arguing *per impossibile*. Their commensurability entails that an odd number equals an even number.

[74] This is clear even *linguistically*.

> [to another]. If it is not equal, it is unequal and [therefore] *contains*, even if it happens not to be commensurable.[75]

Infinite numbers generally do no violence to the unit as indivisible, though there *is* an incompatibility between them with respect to *the measure and the measured*. We must be careful, however, not to limit arithmetic too severely, for in speaking of infinite numbers, we are referring to those which are *actually* so and not to the *open-endedness* of number itself:

> [Aristotle] says that, since magnitude is divided *ad infinitum,* we can always think of a number greater than a given number. For, clearly, division causes multitude. As much as a magnitude is divided, so much does a multitude increase, such that an infinite addition of numbers follows upon the infinite division of magnitudes. But just as the division of magnitude is not infinite in act but in potency and exceeds everything determinately small, as was said, so the addition of numbers is not infinite in act but in potency and exceeds every determinate multitude. But this number thus multiplied to infinity is not a number separate from the division of magnitudes.[76]

[75] *In Metaphysicorum*, 1021. "Cum ergo dicitur in quantitatibus, quod haec est maior illa, vel se habet ad illam ut continens ad contentum, non solum haec ratio non attenditur secundum aliquam determinatam speciem numeri, sed nec etiam quod sit secundum numerum, quia omnis numerus est alteri commensurabilis. Omnes enim numeri habent unam communem mensuram, scilicet unitatem. Sed continens et contentum non dicuntur secundum aliquam commensurationem numeralem. Continens enim ad contentum dicitur, quod est tantum, et adhuc amplius. Et hoc est indeterminatum, utrum sit commensurabile, vel non commensurabile. Quantitas enim qualiscumque accipiatur, vel est aequalis, vel inaequalis. Unde, si non est aequalis, sequitur quod sit inaequalis et continens, etiam si non sit commensurabilis."

[76] *In Physicorum*, III 7393. "Et dicit quod possumus semper intelligere quolibet numero dato alium maiorem, per hoc quod magnitudo dividitur in infinitum. Manifestum est enim quod divisio causat multitudinem: unde quanto plus dividitur magnitudo, tanto maior multitudo consurgit; et ideo ad infinitam divisionem magnitudinum sequitur infinita additio numerorum. Et ideo sicut infinita divisio magnitudinis non est in actu sed in potentia, et excedit omne determinatum in minus, ut dictum est; ita additio numerorum infinita non est in actu sed in potentia, et excedit omnem determinatam multitudinem. Sed hic numerus, qui sic in infinitum multiplicatur, non est numerus separatus a decisione magnitudinum." *Physica*, III 7 207b10–14. "It is always possible to think of a greater [number]: for the bisections of a magnitude are infinite. Hence [this infinite] is in potency, not in act: and what has been taken away always exceeds every determinate quantity. Yet this number is not separate [from the bisections,] nor does the infinite remain, rather, it is *coming about*, like time and the number of time." ἐπὶ δὲ τὸ πλεῖον ἀεὶ ἔστι νοῆσαι· ἄπειροι γὰρ αἱ διχοτομίαι τοῦ μεγέθους. ὥστε δυνάμει μὲν ἔστιν, ἐνεργείᾳ δ' οὔ· ἀλλ' ἀεὶ ὑπερβάλλει τὸ λαμβανόμενον παντὸς ὡρισμένου πλήθους. ἀλλ' οὐ χωριστὸς ὁ ἀριθμὸς οὗτος [τῆς διχοτομίας], οὐδὲ μένει ἡ ἀπειρία ἀλλὰ γίγνεται, ὥσπερ καὶ ὁ χρόνος καὶ ὁ ἀριθμὸς τοῦ χρόνου.

As Aristotle says, "…*this* number is not separate [from the divisions,] nor does the infinite remain, rather, it is *coming about*, like time and the number of time."[77]

Potential infinity—what we termed above as the *open*-endedness of the numbers—is never actual *at once*. If it did measure the unit (and so were properly a number) it would do so by way of *the process of measuring*, not by way of a measurement complete.[78] We *might*, then, speak of numerical infinity in numbers as *in the process of being* measured by the unit, but, as we saw above, this is not to speak of a completed nature, and thus not something measured by the unit.

To briefly conclude, then, imaginary and complex numbers are obviously compromised by the rejection of the numbers upon which they themselves are based. *i* is excluded *a fortiori* if negative one has been excluded, and complex numbers are in part imaginary.

Does Cantor's insistence upon free constructability finds its correlate in the need for the imagination, as understood by Aristotle and Thomas Aquinas? One difficulty is that Aquinas (at least) does not associate the imagination *with* mathematical freedom. Importantly, both Aristotle and Aquinas distinguish the representational function of the imagination from its creative one, which can be a source of falsehood generally and so, too, in mathematics.[79] We will have more to say on this point when we consider the mathematical imagination in Chap. 8.

[77] *Physica*, III 7 207b13–14. (See the reference, immediately above.)

[78] Cantor himself describes the potential infinite similarly: "As to the mathematical infinite, it seems to me that in so far as there has been found a justifiable application for it in the sciences and it has been employed for their advancement, it has appeared principally in the role of a variable which grows beyond all limits or diminishes to any desired minuteness, but which always appears as a quantity remaining finite. This infinite I call *the ideal-infinite* (Uneigenlich-unendlich)." Cantor, *On Infinite, Linear Point-Manifolds*, 5 § 1, p. 92. Translator George Bingley is not entirely comfortable with calling this infinity *the ideal-infinite*. It is on account of its being always finite that there are various renderings of it, such as *the false infinite* (Robin Small), *the improper infinite* (Joseph Dauben), or even *the non-actual infinite* (Bingley himself).

[79] See Marie George, "Imagination a Source of Falsehood According to Aquinas", *Proceedings of the American Catholic Philosophical Association* 67 (1993), especially 193–4, 196.

The Art of the Fiction

In the 1960s and 1970s, Paul Benacerraf famously issued a strong challenge to mathematical realism: one can posit either the extra-mental existence of numbers or a consistent account of how we know them—not both. Benacerraf's own proposal was *structuralism*.[80] Thomist philosophers Thomas Anderson and Armand Maurer (we will turn to them directly in Chap. 13) seem to focus upon the *objects* of mathematics but not mathematical *truth*. They need to face the *epistemological* horn of Benacerraf's dilemma. In other words, given the freedom they afford to mathematical constructions, *how* might we establish that some are false and others true? Or are all true, since mathematical judgments are sufficiently resolved in being imaginable?

As we noted above, Aquinas' threefold distinction (from the *Sentences* passage) is of ways in which our judgments can be true or false. If a remote connection to things existing outside the mind suffices for truth and all imaginables possess such a connection *ipso facto*, then all would *ipso facto* suffice as the basis for true judgments. But the claims made in some (otherwise consistent) mathematical systems in fact contradict the claims made in another.[81] One might further insist that subsequent developments not contradict mathematical concepts *already established*.[82] Still, are the restrictions Aristotle and Aquinas would seemingly place upon mathematical inquiry *justified*? Our ready conclusions, above, would suggest that they are, if only because they are both internally consistent *and* consistent with those remote realities upon which the mathematical sciences are based—the unit and continuous quantity in three dimensions. The restrictions at which we have arrived are not unnecessarily narrow, nor do they restrict the freedom of mathematicians, that is, of persons pursuing mathematical truth through *episteme*.

What of those areas of modern mathematics which *are* demonstrably inconsistent with the first principles we have mentioned? Again and again recent mathematicians have pointed to how successful such advances have

[80] See Benacerraf, "What Numbers Could Not Be" and "Mathematical Truth".

[81] For example, Thomas Anderson explicitly approves of non-Euclidean geometries and would presumably also approve of Euclidean geometry.

[82] *A là* Cantor. Note, though, that if *established* means *imagined*, then the *order* of discovery is contingent and moot.

been—it is patently obvious that modern life would have been impossible absent such systems of thought, to the point that some think it a kind of diminishment to even *raise* questions about their objective truth. Why does it matter whether *i* is adequate to *episteme* since it has been shown to be indispensable in other ways? Surely there must be some truth to be found among the many uses to which modern mathematics is put?

Three things can be said. First, if *episteme* is ever to be found here it will be in virtue of conformity to some real principle. We have shown that that principle is not the mathematical unit or continuous quantity in three dimensions. Second, even if knowing them *cannot* rise to the level of *episteme*, there may be other sorts of truth to be found—not all speculative truths are sciences nor are all truths speculative. Third, even if these imaginative constructions are (as we maintain) pure *artifacts* and do not have natures in the sense provided here, there must be *some* reason why a consistent and powerful body of conclusions can be drawn concerning them. Much of this may have to do with consistency, so important to mathematics generally. Still, even here, such constructs are not mere fictions[83]—we are not free to decide what they will yield once we have conceived them. An account of Sherlock Holmes in which the man was a genuine fool might be called *inconsistent*, but its assumption would not imply that 1 = 2, whereas that *is* the case with many of the systems about which we have been speaking.

We next turn to Aquinas' development of the need for matter in mathematics. Aristotle's *intelligible matter* became a crucial consideration as we responded to the question whether the mathematicals we introduce and study by way of constructive demonstrations are merely *possibilities* and not realities themselves. Since matter is being *potentially*, the ways in which *the potential is real* also became a matter of concern. One of those ways goes by the name of *virtual existence*.

[83] Fictionalism, a recent school of thought in the philosophy of mathematics, is usually seen as having been introduced by Hartry Field in *Science Without Numbers* (Princeton: Princeton University Press, 1980). In saying that we must disallow mathematical developments such as negatives, irrationals, and transfinites, we are not necessarily espousing *this* view. As we pointed out, there *may* be some first principle with which such constructions are compatible. As it is, we have only argued that they are incompatible with the mathematical unit and continuous quantity in three dimensions.

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CHAPTER 7

To Be Virtually

Virtual Reality

In Chap. 4, we saw Aristotle respond to Meno's dilemma by distinguishing various ways in which something can be known and foreknown:

> I think nothing prevents one from, on the one hand, knowing, and on the other, not knowing, what he is learning; for it would be absurd, not if he knew what he was learning in some way, but if he knew it in the very same way as he was learning it.[1]

In his *Commentary on the Posterior Analytics*, Aquinas carefully unwraps Aristotle's argument, speaking of what can be known actually, or universally, or *virtually*:

> [Aristotle] supplies a true solution to this problem in light of the truth already established, saying that nothing prevents one from in some way knowing and in some way not knowing from the very beginning. For it is not absurd if someone in some way already knows what he learns, though it would be, if he already knew it in the same way that he knows it once he has

[1] *Analytica Posteriora*, I 1 71b5–8. ἀλλ' οὐδέν (οἶμαι) κωλύει, ὃ μανθάνει, ἔστιν ὡς ἐπίστασθαι, ἔστι δ' ὡς ἀγνοεῖν· ἄτοπον γὰρ οὐκ εἰ οἶδέ πως ὃ μανθάνει, ἀλλ' εἰ ὡδί, οἷον ᾗ μανθάνει καὶ ὥς.

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> learned it. For, properly speaking, learning occurs when knowledge comes to be. But what comes to be was neither altogether a nonbeing nor altogether a being at the outset, but in a certain way a being and in a certain way a nonbeing: for it was a being in potency, though actually a nonbeing. And this is what it means to come into being, namely, to be reduced from potency to act. Similarly, what someone learns was neither altogether known at first (as Plato said) nor altogether unknown (as those whose solution was refuted above). Rather, it was potentially or *virtually* known in universal principles already understood, though actually unknown as regards a proper understanding. And this is what it means to learn, namely, to be brought from potential or *virtual* or universal knowledge to proper and actual knowledge.[2]

How does knowing something *virtually* differ from knowing it *potentially* and *actually*? *That* they do differ is clear, as to know in one sense is *not to know* in another.

Intuitionism and the Excluded Middle

We learned above that Euclid constructively demonstrates *only five* regular geometrical solids—there *can be* no others—evidence of Aristotle and Aquinas' claim that such conclusions (*decidedly one* of *only two* logical possibilities) are *already* real, *already* true, *already* known, in some way.

Is the *law of excluded middle* necessarily true? This question became a matter of dispute among philosophers and mathematicians of the late nineteenth and early twentieth centuries. *Intuitionists*, who require that all

[2] *In Posteriorum*, I 3 27. "…ponit veram solutionem dubitationis praedictae secundum praedeterminatam veritatem, dicens quod illud quod quis addiscit, nihil prohibet primo quodammodo scire et quodammodo ignorare. Non enim est inconveniens si aliquis quodammodo praesciat id quod addiscit; sed esset inconveniens si hoc modo praecognosceret, secundum quod addiscit. Addiscere enim proprie est scientiam in aliquo generari. Quod autem generatur, ante generationem neque fuit omnino non ens neque omnino ens, sed quodammodo ens et quodammodo non ens: ens quidem in potentia, non ens vero actu: et hoc est generari, reduci de potentia in actum. Unde nec id quod quis addiscit erat omnino prius notum, ut Plato posuit, nec omnino ignotum, ut secundum solutionem supra improbatam ponebatur; sed erat notum potentia sive virtute in principiis praecognitis universalibus, ignotum autem actu, secundum propriam cognitionem. Et hoc est addiscere, reduci de cognitione potentiali, seu virtuali, aut universali, in cognitionem propriam et actualem." Emphases mine. Aquinas references two sorts of passage: from potential (or virtual) to actual and from universal to particular.

mathematical demonstrations be constructive, were led to deny it in some instances, while *classical mathematicians* saw it as an inviolable principle of thought.

While the law is variously stated,[3] Aristotle's own formulation is "there cannot be an intermediate between contradictories, but of one subject we must either affirm or deny any one predicate."[4] There could not possibly be a formulation for which neither contradictory is true.

Why deny this? Partly because intuitionism is a reaction to the classical way of looking at infinite collections of things.[5] Classical mathematicians held that all mathematicals—including statements about infinite collections—were either true or false.[6] Yet that assertion belies an assumption about the mathematics—that there *were* such collections—in relation to which propositions made about them are true or false. Intuitionist mathematician Arend Heyting makes the point:

> You ought to consider what Brouwer's program was. It consisted in the investigation of mental mathematical construction as such, *without questions regarding the nature of the constructed objects, such as whether these objects exist independently of our knowledge of them.*[7]

The law of excluded middle does not apply to subjects whose very *existence* is in question. Mathematics does not wait upon metaphysics. Heyting supplies an example to clarify. We paraphrase.

[3] In modern logic, we say that (p v ~p) is always true. We take (p v ~p) to mean that either a given proposition or its negation must be true, or that there is no third proposition between the two which is true. Anciently, the law of excluded middle was sometimes called *tertium non datur*, literally, *no third [alternative] is given.*

[4] Metaphysica, 4 7 1011b23–5. ἀλλὰ μὴν οὐδὲ μεταξὺ ἀντιφάσεως ἐνδέχεται εἶναι οὐθέν, ἀλλ᾽ ἀνάγκη ἢ φάναι ἢ ἀποφάναι ἓν καθ᾽ ἑνὸς ὁτιοῦν. One must make a distinction between the negation of modern logic and the opposition of Aristotelian logic. For both sorts of logic, the principle of excluded middle is taken to pertain to contradictories, though Aristotelians would maintain that propositions can be opposed in other ways as well. For example, it is not the case that one or the other of two contrary propositions must be true.

[5] Cf. our consideration of Cantor and his contemporaries, above.

[6] More precisely, either a mathematical statement or its negation is true. Illustrative treatments of the three major mathematical schools of modern times can be found in Benacerraf and Putnam, *Philosophy of Mathematics.* Hilbert's optimistic plan for mathematics was the target of the more conservatively minded intuitionists.

[7] Heyting "Disputation," 66. Emphasis mine. L.E.J. Brouwer founded intuitionist mathematics.

Is there a greatest pair of primes differing by one? Here we can easily see whether the statement 'some pair of primes differing by one is the greatest' or its contradictory is true. Here the law of excluded middle applies: either the predicate 'greatest' is truly affirmed of some pair of primes differing by one or it is not—we also happen to know which is so.[8]

Is there a greatest pair of primes differing by two?[9] We have no answer. Intuitionists conclude that the law of excluded middle fails in this case: *neither the assertion nor the negation that there is such a pair is true*. Why the difference? Unless we assume the extra-mental existence of the integers, intuitionists say, we only resolve mathematical questions through constructive demonstrations. Since the twin primes conjecture is unproven, the law of excluded middle does not apply. Classical mathematicians, for their part, insist that the question is *already* resolved within the infinite class of integers.

Heyting is aware of the typical objection:

> One may object that the extent of our knowledge about the existence or non-existence of a last pair of twin primes is purely contingent and entirely irrelevant in questions of mathematical truth. Either an infinity of such pairs exist … or their number is finite: what does it matter whether we can actually calculate the number?[10]

The objection is reasonable and comes easily to mind. What have our ignorance of a proof—the demonstrated truth or falsity of a given mathematical proposition—to do with the objective nature of the thing? All discoveries were unknown to us at some time. No one would say that *magnetized bodies have fields extending beyond their limits* was only true upon being discovered. Realists—mathematical or otherwise—naturally hold that knowledge is measured by the thing we know and not vice versa.[11] Our *inability* to decide the twin-primes question seems beside the point.

[8] Any prime greater than two is odd and any odd number less one is even. Only two and three meet the conditions since, among evens, only two is prime.

[9] So-called *twin primes*.

[10] Heyting, "Disputation," 67.

[11] As Aquinas says: "Natural philosophy examines the order of things that human reason considers but does not make (we include both mathematics and metaphysics under natural philosophy)." *Sententia Libri Ethicorum* (Rome: Marietti Editori, Ltd., 1969), I 1. Hereafter *In Ethicorum*. "Nam ad philosophiam naturalem pertinet considerare ordinem rerum quem ratio humana considerat sed not facit, ita quod sub naturali philosophia comprehendamus et mathematicam et metaphysicam."

Heyting simply points out that realism is a metaphysical, not a mathematical, claim.

> Your argument is metaphysical in nature. If 'to exist' does not mean 'to be constructed', it must have some metaphysical meaning. It cannot be the task of mathematics to investigate this meaning or to decide whether it is tenable or not.[12]

Intuitionists prescind from such questions—to their credit, they are not *mathematical* questions. They are about things they can constructively demonstrate, nothing more.[13] Since classical mathematicians claim to know something about the greatest pair of twin primes—at least that it is either there or not—they assume the integers already exist, a metaphysical assumption.

Classical mathematicians respond predictably. Suppose it had been demonstrated on January 1 of this year that *no pair of twin primes is the greatest.*[14] Would this not have been true to say, even last year? They see intuitionists failing to distinguish between the necessary and objective truth of a proposition and our (contingent) discovery of that truth. Intuitionists point out that the very distinction presupposes a claim about the mathematicals:

> In the study of mental mathematical constructions, 'to exist' must be synonymous with 'to be constructed'. … A mathematical assertion affirms the fact that a certain mathematical construction has been effected. It is clear that before the construction was made, it had not been made. Applying this remark to your example, we see that before January 1, …it had not been proved [that there is an infinite number of twin primes]. … But this is not what you mean. It seems that in order to clarify the sense of your question you must again refer to mathematical concepts: to some world of mathematical things existing independently of our knowledge, where [the statement, 'there is an infinite number of twin primes'] … is true is some absolute sense.[15]

[12] Heyting, *"Disputation,"* 67. Accusing classical mathematicians of hidden metaphysical assumptions is elegant. By and large, they have little regard for metaphysics.

[13] As *mathematicians.* Intuitionists are obviously not doing mathematics in pointing out the flaws of classical mathematics.

[14] Bear in mind that others would take this to mean that there is an infinite number of them.

[15] Heyting, "Disputation," 67–8.

Intuitionists insist upon constructive proofs—and finite proof procedures[16]—precisely because they do not wait upon metaphysics. For purposes of mathematics, the mathematicals are constructed, and we cannot say anything about what does not (yet) exist. The intuitionists' selective denial of the law of excluded middle is rooted in a *particular* division of the sciences.[17]

Unlike classical mathematicians, there is a strong *indeterminacy* in intuitionists' minds when they ask questions—both as to the question and as to its object.

The dilemma is this: do we deny the law of excluded middle or do we assume the actual existence of an infinity of mathematical objects?[18]

There is a middle position.[19] All mathematical propositions, or their contradictories, are true—the law of excluded middle applies. On the other hand, there is no actually infinite universe—mathematical or otherwise.[20] The solution entails seeing how *virtual* existence differs from *actual* and *potential* existence.

Remote and Proximate Objects

While Aquinas' letter *De Mixtione Elementorum*[21] is primarily concerned with another problem, he uses the idea of virtual existence to resolve it. How are elements present in a compound?[22] Not actually, otherwise the whole would be a collection and not a *true mixture*:

[16] "We find writers insisting, as though it were a restrictive condition, that in rigorous mathematics only a finite number of deductions are admissible in a proof—as if anyone had succeeded in making an infinite number of them." Hilbert, "On the Infinite," 193.

[17] Aristotle also selectively denies the principle of excluded middle. See *De Interpretatione*, 9 9a22–19b4.

[18] Remember that *no pair of twin primes is the greatest* is often taking to mean that there actually is an infinite number of them.

[19] Luckily, excluded middle applies to contradictories, not to just any conflicting mathematical views.

[20] Cf. *Physica*, III 4–8.

[21] Thomas Aquinas, "De Mixtione Elementorum ad Magistrum Philippum" in *Opuscula Philosophica* (Rome: Marietti Editori Ltd., 1954), 155–6, hereafter *De Mixtione*. For a parallel account see *Summa Theologiae*, Ia 76 4 obj 4 and *ad* 4.

[22] "Many have difficulty with how elements are present in a mixture." Ibid., 430: "Dubium apud multos esse solet quomodo elementa sunt in mixto."

> So, the different parts of matter subsisting under the forms of the elements take on the notion of many bodies; but it is impossible for many bodies to be in a single place. So, the four elements will not be in every part of the mixed body, and thus it will not be a true mixture (*vera mixtio*) but [only] according to sense, as happens with a collection of invisible or insensible bodies on account of their smallness.[23]

On the other hand, the elements are not simply absent, or there only *potentially*—all substances are potentially in all other substances due to prime matter—after all, a compound is *made of* elements and can be *reduced to* them:

> It seems to some that, since the active and passive qualities of the elements are brought to a mean in some way through alteration, the substantial forms must remain: for if they did not remain, it would seem to be a corruption of the elements, not a mixture.[24]

Moreover, it is of the nature of *elements* to be *in* the substances they make up:

> Further, if the substantial form of a mixed body is the act of matter that does not already have the forms of the simple bodies, then the simple bodies would lose the notion of elements. For an element is that out of which a thing is first made, *and which is in that thing*, and which is indivisible in species; for with the substantial forms [of the elements] lost, a mixed body would not be made up of simple bodies *such that they would remain in it*.[25]

[23] Ibid., 431: "Diversae igitur partes materiae formis elementorum subsistentes plurium corporum rationem suscipiunt. Multa autem corpora impossibile est simul esse. Non igitur in qualibet parte corporis mixti erunt quatuor elementa; et sic non erit vera mixtio, sed secumdum sensum, sicut accidit in congregatione invisibilium sive insensibilium corporum propter parvitatem."

[24] Ibid., 430: "Videtur autem quibusdam quod, qualitatibus activis et passivis elementorum ad medium aliqualiter deductis per alternationem, formae substantiales elementorum maneant: si autem non remaneant, videtur esse corruptio quaedam elementorum, et non mixtio."

[25] Ibid., 431: "Rursus, si forma substantialis corporis mixti sit actus materiae non praesuppositis formis simplicium corporum, tunc simplicia corpora elementorum amittent rationem. Est enim elementum ex quo componitur aliquid primo, et est in eo, et est indivisibile secundum speciem; sublatis enim formis substantialibus, non sic ex simplicibus corporibus corpus mixtum componetur, quod in eo remaneant." Emphasis mine. *In de generatione* adds to this an internal reference to the definition of *element* given in *Metaphysica* V.

The substances of the elements must, then, *be* there in some way beyond potentially and less than actually. Aquinas concludes that elements are *virtually* in compounds. "Therefore, the forms of the elements are not in a mixed body actually, but virtually."[26] What exists virtually does not exist in the primary sense of the word, that is, *actually*. In this respect it is like potential existence. Still, it is not *merely* potential.[27] The potentiality of what exists virtually has been disposed—determined even—to one possible actuality among others. In Aquinas' words, the potentiality is already *determinatur ad unum*.

Similarly, in *De veritate* Aquinas compares the human mind to the potentiality of matter:

> Since, in itself, the possible intellect is in potency to all intelligible forms, just as prime matter is [in potency] to all sensible forms, the intellect is in itself no more determined to assent to the composition [affirmative judgment] than to the division [negative judgment], or conversely.[28]

[26] Ibid., 438–9: "Sunt igitur formae elementorum in mixtis non actu, sed virtute."

[27] Aquinas argues that, since the elemental properties are present in a mean quality which constitutes the proper disposition of the compound, and since the qualities constituting that mean act *in virtue of* the elemental forms, those forms must also be present in the mixture—and more than merely potentially. As he says: "…the active and passive qualities of the elements are contrary to each other, and admit of more and less. But from contrary qualities admitting of more and less a mean quality can be constituted … which is the proper quality of a mixed body … and this quality is the proper disposition to the form of a mixed body, just as a simple quality is to the form of a simple body. Therefore, just as the extremes are found in the mean, which participates in the nature of each, so the qualities of the simple bodies are found in the proper quality of a mixed body. Though the quality of a simple body is other than its substantial form, still, its acts in virtue of that substantial form." Ibid., 438: "…qualitates activae et passivae elementorum sunt ad invicem contrariae, et suscipiunt magis et minus. Ex contrariis autem qualitatibus suscipientibus magis et minus constitui potest media qualitas … quae est propria qualitas corporis mixti, sicut qualitas simplex ad formam corporis simplicis. Sicut igitur extrema inveniuntur in medio, quod participat utriusque naturam, sic qualitates simplicium corporum inveniuntur in propria qualitate corporis mixti. Qualitas autem corporis simplicis est quidem aliud a forma substantiali ipsius, agit tamen in virtute formae substantialis."

[28] Thomas Aquinas, *Quaestiones Disputatae de Veritatae* (Rome: Marietti Editori, Ltd., 1949), 14 2 *respondeo*, hereafter *De Veritate*. "Intellectus autem possibilis, cum, quantum sit de se, sit in potentia respectu omnium intelligibilium formarum, sicut et materia prima respectu omnium sensibilium formarum; est etiam, quantum est de se, non magis determinatus ad hoc quod adhaereat compositioni quam divisioni, vel e converso."

Nothing pre-inclines us to one or the other of contradictory assertions: our mind is initially open to both. But what has no intrinsic tendency of its own (such as prime matter or the intellect itself) must be determined *ab extra*:

> Now, everything which is undetermined in relation to two things is only determined to one of them through something that moves it. But only two things move the possible intellect: its own object, which is an intelligible form or *quiddity* (as is said in *De Anima III*), and the will, which moves all the other powers, as Anselm says. In this way, then, our possible intellect is related differently to the [two] parts of a contradiction.[29]

There are several possibilities regarding the *determinatio* of the intellect to one or the other contradictory, the first of which is the current state of the twin primes conjecture:

> For, sometimes it [the intellect] does not incline more to one than to the other, either because of a lack of moving principles, as in those problems for which we have no reasons [either way], or because of an apparent equality between the [reasons] for each side. This is the condition of one in doubt, who wavers between the two parts of a contradiction.[30]

One among the ways in which the intellect *is* determined to one side of a contradiction is *scientia*.[31] What is the place of *virtual knowledge* in demonstration? Aquinas speaks in many places of things being *virtually*

[29] Ibid. "Omne autem quod est determinatum [Leonine: indeterminatum] ad duo, non determinatur ad unum eorum nisi per aliquid movens ipsum. Intellectus autem possibilis non movetur nisi a duobus; scilicet a proprio obiecto, quod est forma intelligibilis, scilicet quod quid est, ut dicitur in III De Anima, et a voluntate, quae movet omnes alias vires, ut dicit Anselmus. Sic igitur intellectus noster possibilis respectu partium contradictionis se habet diversimode."

[30] Ibid. "Quandoque enim non inclinatur magid ad unum quam ad aliud, vel propter defectum moventium, sicut in illis problematibus de quibus rationes non habemus; vel propter apparentem aequalitatem eorum quae movent ad utramque partem. Et ista est dubitantis dispositio, qui fluctuat inter duas partes contradictionis."

[31] The others are *opinion*, a partial determination accompanied by fear that the opposite claim is true, *understanding*, a complete determination consisting in our immediate apprehension of first principles, and *belief*, an unwavering determination effected by something sufficient to move the will, but not the intellect, of the one who believes.

contained in their principles.[32] Compounds virtually contain the substantial forms of their elements and first principles virtually contain their first conclusions. In neither case can we say that they do so *actually* for, *a*s to the latter, knowing the principles is obviously not the same as knowing the conclusions which follow from them. On the other hand, to insist that the conclusions are *merely* potentially so contained amounts to saying say that the first principles could imply the contradictory. But *scientia* is about necessary conclusions, not contingent ones.[33] It follows that conclusions are *there*, determinately contained in first principles, though not actually.[34]

Aquinas elaborates upon this notion of virtual existence in a question about how angels understand. While immaterial, intellectual substances do not understand things as we do, by affirming and denying, he does discover a similarity between the *human* intellect when it reasons and when it formulates such judgments:

> Just as the conclusion is compared with the principle in the reasoning intellect, so is the predicate with the subject in the intellect composing and dividing. For if our intellect were immediately to see the truth of the conclusion in the principle, it would never understand by discursion or by reasoning. Similarly, if the intellect in apprehending the *quiddity* of the subject were immediately to grasp all that can be attributed to or removed from the subject, it would never understand by composing or dividing but only by understanding the *quiddity*. Clearly, then, it is for the same reason that our intellect understands by discursion as by composing and dividing: in first

[32] A few references from the *Summa Theologiae* alone are Ia 77 8 and 93 7, and II-IIae 17 9 *ad* 3. He instantiates the principle in (1) the relation between the intellectual soul and the sensitive and vegetative souls at Ia 76 3 and 76 4 obj 4 and *ad* 4, (2) the relation between effect and cause at Ia 4 2 and I–IIae 20 5 obj 1 and 88 5 *ad* 2; and (3) the relation between conclusions first principles at Ia 1 7 and 94 3, and I–IIae 3 6.

[33] Contingency is a simultaneous potency to opposites. Cf. *Metaphysica*, 5 5 1015b6–8 and *Summa Theologiae*, Ia 86 3 and IIIa 3 5.

[34] To illustrate, one cannot immediately infer *all C is B* from *all A is B* nor from *all C is A*. The conclusion remains unknown in the light of knowing *either* premise. When we conjoin the premises (through discerning the middle term) we *thereby* see the conclusion. Yet it is not as though those two premises might have been used to deduce anything else—*some C is not B* was *never* a possibility. There is no contingency here.

> apprehending something, it cannot immediately see all that is virtually contained in it.[35]

Why do we not see in a single glance all that the essence of a thing contains but must formulate propositions whereby we slowly come to understand it better? Our initial apprehension of a nature is not so piercing as to enable us to see what is virtually there. Similarly, if we were able to see at a glance all that follows from first principles, we would not need to deduce the conclusions separately. As Aquinas points out in another place:

> …the precepts are to the law what propositions are to the speculative sciences, in which the conclusions are virtually contained in the first principles. So, whoever perfectly and fully [*secundum totam suam virtutem*] understands the principles need not have the conclusions put to him later. But since not all who understand the principles are able to take into account everything virtually contained in them, it is necessary, for their sake, that conclusions be deduced from their principles in the sciences.[36]

The question is not *whether* a given proposition is demonstrated from first principles, then, but whether there is a need to do so in order for it to be true—given the truth of *all A is B* and *all C is A, all C is B* is *already* true. Aquinas tends to side with classical mathematicians in this debate.[37]

[35] *Summa Theologiae*, Ia 58 4 *respondeo.* "Sicut in intellectu ratiocinante comparatur conclusio ad principium, ita in intellectu componente et dividente comparatur praedicatum ad subiectum. Si enim intellectus noster statim in ipso principio videret conclusionis veritatem, nunquam intelligeret discurrendo vel ratiocinando. Similiter si intellectus statim in apprehensione quidditatis subiecti haberet notitiam de omnibus quae possunt attribui subiecto, vel removeri ab eo; nunquam intelligeret componendo et dividendo, sed solum intelligendo quod quid est. Sic igitur patet quod ex eodem provenit quod intellectus noster intelligit discurrendo, et componendo, et dividendo; ex hoc scilicet quod non statim in prima apprehensione alicuius primi apprehensi potest inspicere quidquid in eo virtute continetur."

[36] Ibid., II–IIae 44 2 *respondeo.* "Hoc modo se habent praecepta in lege, sicut propositiones in scientiis speculativis; in quibus conclusiones virtute continentur in primis principiis. Unde qui perfecte cognosceret principia secundum totam suam virtutem, non opus haberet ut ei conclusiones seorsum proponerentur. Sed quia non omnes qui cognoscunt principia, sufficiunt considerare quidquid in principiis virtute continetur, necesse est propter eos ut in scientiis ex principiis conclusiones deducantur."

[37] Recall that this is the sort of distinction they insist upon in their explanation of how a proposition can be true and yet not known to be true.

Returning to the conjecture, then, let us propose that *no pair of twin primes is greatest*. Classical mathematicians would say we spoke either truly or falsely, while intuitionists would say that our claim is neither true nor false absent a constructive demonstration.[38] What about Aquinas?

The law of excluded middle applies to the mathematical sciences. Since conclusions are virtually contained in their first principles, Aquinas would claim, along with classical mathematicians, that either a given statement or its contradictory is true even before it has been shown to be so by reduction to first principles—which is what demonstration is. *Indeterminacy* in mathematical questions is on the part of the one asking them, not on the part of the object. Unlike what is merely potential, mathematical conclusions are determinately contained in their principles. Given this agreement with classical mathematicians, is Aquinas required thereby to posit an infinite class of integers existing outside the mind?

Not so—and in this Aquinas agrees with the intuitionists. For the answer is virtually contained in the principles of arithmetic, which is to say that the opposite *could not* be proved, or that the principles are already *determinatur ad unum* with respect to that *specific disjunction*. Though we cannot see everything contained in mathematical first principles, still, what is contained therein is *determinately* contained, in a manner sufficient also to determine our minds when drawing a conclusion. As for the conjecture, given the nature of the mathematical unit,[39] the property of being *greatest* is either compatible in part or universally incompatible with *twin prime*. Given our intellectual frailty—that we cannot see in a glance what the principles require in this instance—what remains is to discover those connections required to reach a conclusion. Yet that is a merely contingent event. It could occur now, or never.

Early in the *Metaphysics*, Aristotle takes up the views of his predecessors, from the Pre-Socratic philosophers through the Pythagoreans and Plato. In the course of objecting to the Pythagorean view of numbers, Aristotle

[38] Intuitionists would seem unable to deal with negatives, like the *nongreatest prime,* which cannot be constructed. But apparently some even regard indirect proof, that is, showing that a construction yields absurd or impossible results, as a sort of construction. Cf. Stephan Korner, *The Philosophy of Mathematics* (New York: Harper & Brothers, 1960), 38.

[39] Again, the unit is the first principle of arithmetic.

mentions that the unit is, in a way, both even and odd.[40] This is very puzzling, until one takes into account his discussion of Meno's dilemma, as well as Aquinas's subsequent division of knowledge into actual, potential, and virtual. Aquinas makes the connection explicit:

> The unit is virtually both even and odd. For all differences of number are virtually found in the unit, since all differences of number are resolved into the unit. Hence, the unit is first among odd numbers. So, too, with even, square, perfect, and the other differences of number: because, *though the unit is not actually a number, it is still virtually every number*. And just as the one was said to be composed of the even and odd, likewise a number is composed of units.[41]

To the questions what is mathematics about? Aquinas would say, continuous quantity in three dimensions and the unit, as well as those constructs which necessarily follow from them.[42] Let us recall a passage we introduced above, which sheds light on this issue as well:

> In those sciences which are about certain accidents, nothing prevents that which is taken as a subject with respect to a certain attribute to be also taken as an attribute with respect to a prior subject. Still, this cannot go on infinitely: for one must come to something which is first in that science, which is taken as a subject and in no way as an attribute; as is clear in the mathematical sciences, which are about continuous or discrete quantity. For those things which are first in the genus of quantity are assumed in these sciences:

[40] *Metaphysica,* I 5 986a18–20. "[The Pythagoreans] hold that the elements of number are the even and the odd—the one limited and the other unlimited—and that the unit comes from both of them (for *the unit is both even and odd*) and number comes from the unit." Emphasis mine. τοῦ δὲ ἀριθμοῦ στοιχεῖα τό τε ἄρτιον καὶ τὸ περιττόν, τούτων δὲ τὸ μὲν πεπερασμένον τὸ δὲ ἄπειρον, τὸ δ᾽ ἓν ἐξ ἀμφοτέρων εἶναι τούτων (καὶ γὰρ ἄρτιον εἶναι καὶ περιττόν), τὸν δ᾽ ἀριθμὸν ἐκ τοῦ ἑνός.

[41] *In Metaphysicorum*, I 8 126. "Unitas enim et par est virtute et impar. Omnes enim differentiae numeri unitati conveniunt in virtute, quia quaecumque differentiae numeri in unitate resolvuntur. Unde in ordine imparium primum invenitur unitas. Et similiter in ordine parium et quadratorum et perfectorum numerorum, et sic de aliis numeri differentiis: quia unitas licet non sit actu aliquis numerus, est tamen omnis numerus virtute. Et sicut unum dicebat componi ex pari et impari, ita numerum ex unitatibus." Emphasis mine.

[42] Both Aristotle and Aquinas describe the *generation* of numbers from the unit by way of a division of continuous quantity but is not clear whether this process is *required for* our apprehension of the arithmetical unit and some of the numbers. *Some* sort of construction is obviously necessary for the higher numbers. It may be that an abstractive apprehension of *continuous quantity in three dimensions* is sufficient for the mathematical sciences.

> for example, the unit, and the line and surface, and others of this sort. Assuming these, certain other things are sought through demonstration, such as the equilateral triangle and the square in geometry, and other such things. This is why such demonstrations are called operative, as it were, for example: on a given line to construct an equilateral triangle. This achieved, certain attributes are then proved about it, for example, that its angles are equal, or some such thing. It is clear, therefore, that in the first manner of demonstration triangle is an attribute, while in the second it is a subject. So the Philosopher is using triangle as an attribute here, and not as a subject, when he says "of triangle it is necessary to already know what it means."[43]

Mathematics is about things we make, yes, but also (and ultimately) about things we discover. No surprise that a discipline noted for logical rigor in demonstration could draw out so much from such simple principles.

What sort of realism does Aquinas propose, then? If the unit is all numbers virtually, then the arithmetic—howsoever much of it remains to be discovered—is already *decided*. The sorts of number which can be legitimated upon its basis are *already* limited. So, too, for the geometry of continuous quantity in three dimensions.[44] Mathematics is not so much *free* as already virtually complete. This realism is entirely consistent with Aristotle's claim that conclusions (which is to say, determinately one of only two logical possibilities) are *already known*, in some way. They are present more than *purely* materially (contingent possibilities) but not actually.

[43] *In Posteriorum Analyticorum*, I 2 17. "In illis autem scientiis, quae sunt de aliquibus accidentibus, nihil prohibet id, quod accipitur ut subiectum respectu alicuius passionis, accipi etiam ut passionem respectu anterioris subiecti. Hoc tamen non in infinitum procedit. Est enim devinire ad aliquod primum in scientia illa, quod ita accipitur ut subiectum, quod nullo modo ut passio; sicut patet in mathematicis scientiis, quae sunt de quantitate continua vel discreta. Supponuntur enim in his scientiis ea quae sunt prima in genere quantitatis; sicut unitas, et linea, et superficies, et alia huiusmodi. Quibus suppositis, per demonstrationem quaeruntur quaedam alia, sicut triangulus aequilaterus, quadratum in geometricis et alia huiusmodi. Quare quidem demonstrationes quasi operativae dicuntur, ut est illud, Super rectam lineam datam triangulum aequilaterum constituere. Quo adinvento, rursus de eo aliquae passiones probantur, sicut quod eius anguli sunt aequales aut aliquid huiusmodi. Patet igitur quod triangulus in primo modo demonstrationis se habet ut passio, in secundo se habet ut subiectum. Unde Philosophus hic exemplificat de triangulo ut est passio, non ut est subiectum, cum dicit quod de triangulo oportet praescire quoniam hoc significat."

[44] We have already come across one implication of this in Euclid's proof that only five regular solids can be constructed.

Negatively, then, conclusions and constructions demonstrably incompatible with mathematical principles, as we really find them, are unsound. Mathematical questions and conjectures are answerable in principle, since one outcome is decidedly true. This assurance is based on reality and the nature of quantity. Moreover, the line between the truths of mathematics and consistent mathematical fictions can be clearly drawn.

On the other hand, innumerable constructs which are demonstrably incompatible with the unit (such as fractions, negative numbers, and the like) are plain useful. Why should a *fiction* be both consistent and of daily use? We next turn to another aspect of Aristotle's mathematical realism to which Aquinas brought clarity: mathematics as an art. Rooted in Plato's consideration of the education of the guardians of the republic, the mathematical arts have historically come down to us as four in a group of seven—the medieval *quadrivium* among the so-called *liberal arts*.

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CHAPTER 8

Mathematics and the Liberal Arts

Episteme and Art

In distinguishing *episteme* from other intellectual virtues, Aristotle begins with the difference between the *scientific* part of thinking and the *calculative*.

> Let it be supposed that there are two parts having reason: one by which we contemplate things whose causes cannot be otherwise, the other by which we contemplate things [whose causes] can: for according to this difference in kind the parts of the soul naturally also differ in kind, since each has knowledge *on account of a certain similarity and conformity*.[1]

We have already dealt with how the inquiring mind is conformed to the known object in the scientific part—as Aristotle puts it: "It is not because we truly think that you are pale that you are pale, but because you are pale we, who say this, speak truly."[2] If, as he suggests here, truth and

[1] *Ethica Nicomachea*, VI 1 1139a6-11. Emphasis mine. καὶ ὑποκείσθω δύο τὰ λόγον ἔχοντα, ἓν μὲν ᾧ θεωροῦμεν τὰ τοιαῦτα τῶν ὄντων ὅσων αἱ ἀρχαὶ μὴ ἐνδέχονται ἄλλως ἔχειν, ἓν δὲ ᾧ τὰ ἐνδεχόμενα: πρὸς γὰρ τὰ τῷ γένει ἕτερα καὶ τῶν τῆς ψυχῆς μορίων ἕτερον τῷ γένει τὸ πρὸς ἑκάτερον πεφυκός, εἴπερ καθ᾽ ὁμοιότητά τινα καὶ οἰκειότητα ἡ γνῶσις ὑπάρχει αὐτοῖς.

[2] *Metaphysica*, IX 10 1051b7-9. οὐ γὰρ διὰ τὸ ἡμᾶς οἴεσθαι ἀληθῶς σε λευκὸν εἶναι εἶ σὺ λευκός, ἀλλὰ διὰ τὸ σὲ εἶναι λευκὸν ἡμεῖς οἱ φάντες τοῦτο ἀληθεύομεν.

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conformity go hand-in-hand, how does it apply when we use our minds in contingent matters, *practically*?

Calculative thought (art and prudence) *measures* things, not vice versa. Not that scientific knowledge can in no way be a measure,[3] but that it is not so principally:

> We call science (ἐπιστήμη) and sense-knowledge measures of things for the same reason, that we know something through them; yet they are measured more than they measure.[4]

Aquinas elaborates upon knowledge as a *measure*. While ἐπιστήμη and perception are measures only secondarily, practical knowledge measures things, properly speaking:

> If there is knowledge which is the cause of the thing known, it must be the measure of that thing. For example, the knowledge of the artisan is the measure of the artifacts, since any artifact is perfect to the degree that it attains a likeness to the art.[5]

In the introduction to his *Commentary on the Nicomachean Ethics,* Aquinas famously divides the consideration of the human intellect into four types of order. Knowledge being both scientific and practical, we also need to have regard for what we knowingly produce, beyond what we have discovered.

> Order is related to reason in a fourfold way. For there is one order which reason does not make but only considers … There is another order which reason by considering makes in its own act … There is a third order which reason by considering makes in the operations of the will. And there is a

[3] See our discussion of the *middle sciences*, above.

[4] *Metaphysica*, X 1 1053a31-34. καὶ τὴν ἐπιστήμην δὲ μέτρον τῶν πραγμάτων λέγομεν καὶ τὴν αἴσθησιν διὰ τὸ αὐτό, ὅτι γνωρίζομέν τι αὐταῖς, ἐπεὶ μετροῦνται μᾶλλον ἢ μετροῦσιν. Scientific truth is a measure only *after* the mind has been measured by things.

[5] *In Metaphysicorum,* X 2 1959. "Si qua vero scientia est quae est causa rei scitae, oportebit quod sit eius mensura. Ut scientia artificis est mensura artificiatorum; quia unumquodque artificiatum secundum hoc perfectum est, quod attingit ad similitudinem artis." We have rendered *scientia* broadly here so as to include art, which is properly *distinguished from* science.

> fourth order which reason by considering makes in those exterior things of which it is the cause.[6]

The first type is the *scientific* order, while the order reason produces in its own thoughts is *logic*, since it is here that the intellect reflects upon its own acts and applies this knowledge to them, that it may reason more surely and without error.[7] The order reason knowingly brings to our acts of will constitutes *ethics* and *politics*, whereby we are able to direct our choices to proper means to good ends. Finally, reason brings order to things we make, which is *art*. Corresponding to each order is an analogous sense of knowledge and truth. [8] Beyond scientific knowledge, logical and artistic knowledge are *factive*,[9] while moral truth requires right desire and right reason.[10] Common to these *knowings* is that they measure other things.

Being reciprocal, *mutual* similarity or correlation do not sufficiently designate a principle or term. It is not even enough to say that truth lies in *conformity between mind and thing*, since this is not to designate the intellect, rather than the thing, as a principle. Unlike scientific truth, the truth of the *calculating* mind will be found in the things we have made or done. *Conformity* is an irreversible relation. Thus Aquinas:

[6] *In Ethicorum*, I 1 1. "Ordo autem quadrupliciter ad rationem comparatur. Est enim quidam ordo quem ratio non facit, sed solum considerat … Alius autem est ordo, quem ratio considerando facit in proprio actu … Tertius autem est ordo quem ratio considerando facit in operationibus voluntatis. Quartus autem est ordo quem ratio considerando facit in exterioribus rebus, quarum ipsa est causa."

[7] "If, because reason reasoned about the act of the hand, the art of building or carpentry was discovered, through which man is able to exercise acts of this sort easily and in an ordered way, for the same reason a certain art, which can direct the act of reasoning itself, is needed, through which a man might proceed in an orderly way, easily, and without error, in the act of reason itself. And this art is logic, that is, rational science." *In Posteriorum, Prooemium* 1–2. "Si igitur ex hoc, quod ratio de actu manus ratiocinatur, adinventa est ars aedificatoria vel fabrilis, per quas homo faciliter et ordinate huiusmodi actus exercere potest; eadem ratione ars quaedam necessaria est, quae sit directiva ipsius actus rationis, per quam scilicet homo in ipso actu rationis ordinate, faciliter et sine errore procedat. Et haec ars est Logica, id est rationalis scientia."

[8] Aquinas masterfully distinguishes various senses of truth in *De Veritate*, 1 2.

[9] At least in part. See the distinction between *logica docens* and *logica utens* at the end of this section.

[10] *In Ethicorum*, II 7 322, 326; III 15 548; and VI 2 1130–31.

> A thing is compared to the practical intellect in one way and to the speculative intellect in another. For the practical intellect causes things and so it is a measure of what comes about through it. But the speculative intellect, since it receives from things, is, in a certain way, moved by them, and thus [these] things measure it. Whence it is clear that natural things, from which our intellect receives knowledge, measure our intellect, as is said in the tenth book of the *Metaphysics*: but these [natural things] are measured by the divine intellect, in which are all created things, just as all artificial things in the intellect of the artist [are measured by the artist's intellect].[11]

While art and prudence *rightly* determine means suitable for making or doing, still, practical truth *proper* is the conformity of these to what they *had in mind*. Makings are *true* by virtue of their conformity to the art residing in the mind of the artist. Still, what the artist brings to a making *intellectually* can be true or false as well.

Of the four types of order, mentioned above, thinking about our own thoughts has both a scientific and a practical part. The *scientific* logician thinks about second intentions—what they and their properties are—while an *application* of logic to thought pertains to the (liberal) *art* of logic. This is the basis of Aquinas' distinction between art as *docens* and as *utens*.

> Though philosophy is said to be a science where dialectics and sophistics are not, this does not do away with dialectics and sophistics beings sciences [all the same]. For dialectics can be taken as *docens* or as *utens*. As *docens*, it studies [second] intentions and determines *how* one can demonstrate thereby that the conclusions of the individual sciences *probably* follow. It does this demonstratively, and so is a science to that extent. But as *utens*, it uses the method it has discovered to *arrive at* the probable conclusions of those individual sciences. To that extent it falls short of being a science. The same can be said for sophistics. As *docens*, it thinks about how we can argue to

[11] *De Veritate*, 1 2 *corpus*. "Res aliter comparatur ad intellectum practicum, aliter ad speculativum. Intellectus enim practicus causat res, unde est mensuratio rerum quae per ipsum fiunt: sed intellectus speculativus, quia accipit a rebus, est quadammodo motus ab ipsis rebus, et ita res mensurant ipsum. Ex quo patet quod res naturales, ex quibus intellectus noster scientiam accipit, mensurant intellectum nostrum, ut dicitur X *Metaph.*: sed sunt mensuratae ab intellectu divino, in quo sunt omnia creata, sicut omnia artificia in intellectu artificis."

> *apparent* truth using necessary and demonstrative arguments, whereas as *utens*, it falls short of the way of true argumentation.[12]

What artists *bring to* their makings, intellectually, itself needs to be *right*, even before one considers whether their products are *true*, in the sense explained above. After all, both Aristotle and Aquinas understand art to be "*right* reason in making."[13]

Aquinas turns to the art of demonstration next:

> In the demonstrative part of logic its *teaching* [*doctrina*, cf. *docens*] belongs to logic alone, while its use belongs to philosophy and to other, particular, sciences which are about the nature of things. This is because the use of the demonstrative part consists in using the principles of *things*—not logical intentions—from which [use] proceeds demonstration, which pertains to the real sciences. So it appears that certain parts of logic have science, teaching, and use all together—for example investigative dialectics and sophistics—while others have teaching but not use, namely, demonstrative logic.[14]

The Mathematical Arts

The logical arts have this in common with mathematics: in that they are *liberal* arts.

[12] *In Metaphysicorum,* IV 4 576. Emphasis mine. "Licet autem dicatur, quod philosophia est scientia, non autem dialectica et sophistica, non tamen per hoc removetur quin dialectica et sophistica sint scientiae. Dialectica enim potest considerari secundum quod est docens, et secundum quod est utens. Secundum quidem quod est docens, habet considerationem de istis intentionibus, instituens modum, quo per eas procedi possit ad conclusiones in singulis scientiis probabiliter ostendendas; et hoc demonstrative facit, et secundum hoc est scientia. Utens vero est secundum quod modo adinvento utitur ad concludendum aliquid probabiliter in singulis scientiis; et sic recedit a modo scientiae. Et similiter dicendum est de sophistica; quia prout est docens tradit per necessarias et demonstrativas rationes modum arguendi apparenter. Secundum vero quod est utens, deficit a processu verae argumentationis."

[13] Cf. *Ethica Nicomachea*, VI 4 1040a9–10. "a habit of making in accordance with true reason" ἕξις μετὰ λόγου ἀληθοῦς ποιητική. *In Ethicorum*, VI 3 1153. "habitus factivus cum vera ratione".

[14] *In Metaphysicorum,* IV 4 576. "Sed in parte logicae quae dicitur demonstrativa, solum doctrina pertinet ad logicam, usus vero ad philosophiam et ad alias particulares scientias quae sunt de rebus naturae. Et hoc ideo, quia usus demonstrativae consistit in utendo principiis rerum, de quibus fit demonstratio, quae ad scientias reales pertinet, non utendo intentionibus logicis. Et sic apparet, quod quaedam partes logicae habent ipsam scientiam et doctrinam et usum, sicut dialectica tentativa et sophistica; quaedam autem doctrinam et non usum, sicut demonstrativa." Emphasis mine.

> [The seven liberal arts] are grouped together because they were first taught to those who wanted to learn philosophy. And they were divided into *trivium* and *quadrivium* because by these paths, as it were, a quick mind is introduced to the secrets of philosophy.[15]

Aquinas finds that both this need for training in the arts of logic and the need for subsequently learning mathematics is consonant with Aristotle's own thought:

> This fits with the Philosopher's words in *Metaphysics* II that we must investigate the method of the sciences before the sciences themselves. The Commentator says (in the same place) that one should learn logic—the *trivium*, which is about the methods of all the sciences—before all the other sciences. [The Philosopher] also says in *Ethics* VI that the young can learn mathematics but not natural philosophy, which requires experience. So we are given to understand that after logic we should learn the quadrivium—mathematics. These, then, are like paths preparing the mind for the other philosophical disciplines.[16]

As to why these preliminary studies are called *arts*:

> These, among the other sciences, are also said to be *arts*, not only because they *know*, but because they have some product that is immediately due to reason itself—constructing a syllogism or making a speech, numbering, measuring, composing melodies, or computing the course of the stars. Other sciences (divine and natural science, for example) either do not have

[15] *In De Trinitate*, 5 1 *ad* 3. "[S]eptem connumerantur, quia his primum erudiebantur, qui philosophiam discere volebant, et ideo distinguuntur in trivium et quadrivium, eo quod his quasi quibusdam viis vivax animus ad secreta philosophiae introeat."

[16] Ibid., "Et hoc etiam consonat verbis philosophi qui dicit in II metaphysicae quod modus scientiae debet quaeri ante scientias; et Commentator ibidem dicit quod logicam, quae docet modum omnium scientiarum, debet quis addiscere ante omnes alias scientias, ad quam pertinet trivium. Dicit etiam in VI Ethicorum quod mathematica potest sciri a pueris, non autem physica, quae experimentum requirit. Et sic datur intelligi quod post logicam consequenter debet mathematica addisci, ad quam pertinet quadrivium; et ita his quasi quibusdam viis praeparatur animus ad alias philosophicas disciplinas."

> a product but only knowledge … or they have a physical product, such as medicine, alchemy, and such other sciences.[17]

As we shall eventually conclude, Aristotelian and Thomistic mathematics is a *mixed bag*. While included among the strictly *scientific* intellectual habits, it nevertheless employs properly *constructive* demonstrations, takes the form of a *mixed* science (conjoined with optics or harmonics, for example), and constitutes the *quadrivium* of the liberal arts. In an article on the liberal arts in Aquinas, Pierre Conway and Benedict Ashley work out these various aspects of mathematics through the responses to a series of objections Aquinas presents to his own division of the sciences.

> From this outlook, then, the meaning of St. Thomas, who lists the order of learning as logic, mathematics, natural science, moral science and metaphysics, without any special distinguishing of the first two from the latter three, is that one will first learn logic as an art, i.e., how to construct a sentence, form a syllogism, compose a speech; followed by the learning of mathematics, first as an art, involving measuring, constructing and the like, and then as a science, i.e., by beginning to see, especially through the application of logic, why things are as they are. Thus, through the constructions made in connection with the right triangle of the Pythagorean theorem, and which have the character of art, one is able clearly to show the role of those constructions as a middle term—as known from logic—linking together the square on the hypotenuse with the squares on the other two sides. In so doing, one shows or demonstrates why this must be so, i.e., one induces science.[18]

[17] *In De Trinitate*, 5 1 *ad* 3. "…hae inter ceteras scientias artes dicuntur, quia non solum habent cognitionem, sed opus aliquod, quod est immediate ipsius rationis, ut constructionem syllogismi vel orationem formare, numerare, mensurare, melodias formare et cursus siderum computare. Aliae vero scientiae vel non habent opus, sed cognitionem tantum, sicut scientia divina et naturalis; unde nomen artis habere non possunt, cum ars dicatur ratio factiva, ut dicitur in VI metaphysicae. Vel habent opus corporale, sicut medicina, alchimia et aliae huiusmodi. Unde non possunt dici artes liberales, quia sunt hominis huiusmodi actus ex parte illa, qua non est liber, scilicet ex parte corporis. Scientia vero moralis, quamvis sit propter operationem, tamen illa operatio non est actus scientiae, sed magis virtutis, ut patet in libro Ethicorum. Unde non potest dici ars, sed magis in illis operationibus se habet virtus loco artis. Et ideo veteres diffinierunt virtutem esse artem bene recteque vivendi, ut Augustinus dicit in IV de civitate Dei."

[18] Pierre Conway and Benedict Ashley, "The Liberal Arts in St. Thomas Aquinas," *The Thomist* 22, no. 4 (1959): 490.

If mathematics turns out to be something this complex, it is no wonder that some one aspect might be fixed upon as wholly determinative. Choosing to emphasize its *scientific* (and so *realist*) character, one might be inclined to see its objects—all its objects—as having a place in either the immaterial or material realms; such an emphasis might see the use of constructive definitions as one step toward fictionalism. On the other hand, overemphasizing its *artistic* character might lead to an exaggerated emphasis on free constructability among the mathematicals; other considerations (such as compatibility with mathematical principles) might then be regarded as merely limiting. Finally, focusing upon the real benefits of *mathematica utens*—the middle sciences, for example—could well lead someone to consider success the sole, or at least the overwhelmingly important, criterion of its value; the more practically minded among us might then tend to regard *the real* as irrelevant.

Aquinas' philosophy of mathematics takes all these aspects into account. It is yet one more instance of the *media via* for which he is rightly famous. We will later address how this *unified* mathematics allows Aquinas to avoid some of the more contemporary problems besetting mathematical-philosophers. For now, we will turn to one final way in which Aquinas builds upon the mathematical foundations discovered by his Greek predecessor, namely, his so-called *reduction of mathematics to the imagination*.

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CHAPTER 9

The Place of the Imagination

Imagination: A Thomistic Development

While Plato and Aristotle *employ* mathematical examples by way of illustrating particular philosophical truths (cf. doubling the square, the divided line, the sum of the angles of a triangle, constructing a triangle in a semicircle, and many others), neither left behind a *mathematical work*, like Euclid's *Elements* or Nicomachus of Gerasa's *Introduction to Arithmetic*. Nor did Aquinas, though he did write an *ex professo* treatment of mathematics as part of his *Commentary on the De Trinitate of Boethius.* In Part Two of the *De Trinitate* Boethius, following Aristotle, makes and explains the distinction among the sciences:

> Speculative Science may be divided into three kinds: Physics, Mathematics, and Theology. Physics deals with motion and is not abstract or separable (i.e., ἀναπεξαίρετος); for it is concerned with the forms of bodies together with their constituent matter, which forms cannot be separated in reality from their bodies. As the bodies are in motion—the earth, for instance, tending downwards, and fire tending upwards, form takes on the movement of the particular thing to which it is annexed. Mathematics does not deal with motion and is not abstract, for it investigates forms of bodies apart from matter, and therefore apart from movement, which forms, however, being connected with matter cannot be really separated from bodies. Theology does not deal with motion and is abstract and separable, for the Divine Substance is without either matter or motion. In Physics, then, we

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> are bound to use scientific, in Mathematics, systematical, in Theology, intellectual concepts; and in Theology we will not let ourselves be diverted to play with imaginations but will simply apprehend that Form which is pure form and no image, which is very Being and the source of Being.[1]

While we have already dealt with the distinctions among the *objects* of these, in terms of their separability from matter and movement, the appellations *scientific, systematical, and intellectual* as points of difference are new. Aquinas ascribes such differences, not to *what* we consider in each science, but to *how* we consider it, i.e., to the *methods* of the sciences. As he says:

> Then when [Boethius] says: *In the natural sciences…*, he shows what method is used in each [science]. Here, he does two things. He first establishes which method is used in each of these parts—the explanation of this part remains to be discussed. Then he explains the last method—the point of the present discussion. He does this in two ways, first, by removing an impediment, saying *we need not be led by images to deductions in divine matters*—as if we were to follow the judgment of the imagination in judging of them, secondly, by showing what [method] is proper: *but we ought rather to consider the form itself* apart from motion and matter, whose conditions he consequently explains as he begins the investigation he has proposed.[2]

[1] Boethius, *De Trinitate*, trans. H.F. Stewart and E.K. Rand (Massachusetts: Harvard University Press, 1968), II 5–21. "Nam cum tres sint speculativae partes, naturalis, in motu inabstracta ἀναπεξαίρετος (considerat enim corporum formas cum materia, quae a corporibus actu separari non possunt, quae corpora in motu sunt ut cum terra deorsum ignis sursum fertur, habet que motum forma materiae coniuncta), mathematica, sine motu inabstracta (haec enim formas corporum speculatur sine materia ac per hoc sine motu, quae formae cum in materia sint, ab his separari non possunt), theologica, sine motu abstracta atque separabilis (nam dei substantia et materia et motu caret), in naturalibus igitur rationabiliter, in mathematicis disciplinaliter, in divinis intellectualiter versari oportebit neque diduci ad imaginationes, sed potius ipsam inspicere formam quae vere forma neque imago est et quae esse ipsum est et ex qua esse est."

[2] *In De Trinitate,* 1 2 2 1. "Deinde cum dicit: in naturalibus igitur etc., ostendit, quis sit modus congruus praedictis partibus. Et circa hoc duo facit. Primo concludit modos congruos singulis partium praedictarum, et huius partis expositio relinquitur disputationi. Secundo exponit ultimum modum qui est proprius praesenti inquisitioni. Et hoc dupliciter. Primo removendo id quod est impeditivum dicens: neque oportet in divinis deduci ad imaginationes, ut scilicet de eis iudicando sequamur imaginationis iudicium. Secundo ostendendo id quod est proprium, ibi: sed potius ipsam inspicere formam sine motu et materia, cuius condiciones consequenter exponit ingrediens ad propositam inquisitionem."

The *commentary proper* ends here. In what follows, Aquinas utilizes the *disputed question* form and raises a number of questions and articles regarding Boethius' division of the sciences and their methods.[3] Of these, articles one and two of question six concern us, as they reference Aquinas' view of the place of imagination in mathematics.[4]

Aquinas divides the first article into three parts, each concerned with the proper method of one of the three speculative sciences: natural philosophy, mathematics, and divine science (metaphysics). He is concerned with how these sciences begin, how they proceed, and how they resolve. Following Boethius, Aquinas describes them as proceeding *rationally*, *disciplinabiliter*,[5] and *intellectually*, respectively. As to the method of mathematics, Aquinas has this to say:

> To proceed *disciplinabiliter* is to proceed demonstratively and with certitude. But, as Ptolemy says in the beginning of the *Almagest*, "Mathematics alone—if one brings diligence to that inquiry—will bring a firm and stable belief to the one seeking knowledge [of it], by demonstrating [its conclusions] in unquestionable ways." Therefore, it is most proper to mathematics to proceed *disciplinabiliter*.[6]

This is not because other sciences do not proceed demonstratively, but because mathematical demonstrations exhibit great certitude in their conclusions.[7] For Aquinas, this has to do with the way we think about the

[3] These have come to be separately named *The Division and Methods of the Sciences.*

[4] Though the *Division and Methods of the Sciences* is a *locus classicus* for Aquinas on the mathematical imagination, he frequently associates mathematics with the imagination in his works. Cf. *Scriptum Super Sententiarum,* 3 *dist.* 13, 1 2 3 *ad* 1; *In Aristotelis Librum De Anima Commentaria* (Rome: Marietti Editori, Ltd., 1959), 3 8 16–7, hereafter *In De Anima*; *De veritate* 15 2 *corpus*; *In Physicorum* 3 7 6, 8 5 3; *In Metaphysicorum*, 2 5 4, 7 10 14; *In Ethicorum,* 1 11 7, 6 7 16–17, 20; and *In Aristotelis Librum de Memoria et Reminiscentia Commentarium* (Rome: Marietti Editori, Ltd., 1928), 2 2 2, hereafter *In De Memoria*.

[5] *Rationabiliter* does translate easily into *rationally* and *intellectuabiliter* into *intellectually*, but there is no easy translation for *disciplinabiliter*, except, perhaps, *in the manner of learning—learningly* is simply monstrous.

[6] *In De Trinitate*, 6 1 *sed contra* 2. "Sed contra, disciplinaliter procedere est demonstrative procedere et per certitudinem. Sed, sicut Ptolemaeus in principio Almagesti dicit, solum mathematicum genus, si quis huic diligentiam exhibeat inquisitionis, firmam stabilemque fidem intendentibus notitiam dabit, velut demonstratione per indubitabiles vias facta. Ergo disciplinaliter procedere maxime proprium est mathematici."

[7] Cf. Ibid., *corpus* 2.

mathematicals—abstracting them from their sensible conditions (unlike natural philosophy) but not to the point of dissociating them from sensible reality altogether (like metaphysics).

In the second article of question six, Aquinas speaks of the method of divine science—in particular, as to whether images are of use here. More the contrary, as it turns out.

> In all knowledge two things must be considered: the beginning and the end. The beginning pertains to apprehension and the end to judgment, for knowledge is completed [in judgment].[8]

Following Aristotle's account of *episteme*, Aquinas traces the steps needed to arrive at certitude in our demonstrated conclusions. In reverse order, conclusions are judgments (propositions) known to be true on the basis of known premises, which are also judgments either known to be themselves true on the basis of true premises, or (ultimately) known to be true *in themselves*. In *Posterior Analytics* II 13, Aristotle ultimately grounds our knowledge of such first principles in sensation. The *beginning* of all our knowledge, then, is sensation, though—unlike strict empiricists—Aristotle and Aquinas deny that our knowledge goes no *further than* sensation (as is clear from their division of the sciences, which include metaphysics). As to this beginning, Aquinas continues.

> The beginning of all our knowledge is in the senses, since from the apprehension of the senses arises the apprehension of the imagination—which is a movement made [in us] by the senses, as the Philosopher says—from which, in turn, arises our intellectual apprehension, since images are like the objects of the intellective soul, as is clear in *De Anima* III.[9]

In each case—sensation, imagination, intellection—Aquinas speaks of an *apprehension*, that is, a *taking in what is there*. Importantly,

[8] Ibid., 6 2 *corpus*. "Responsio. Dicendum quod in qualibet cognitione duo est considerare, scilicet principium et terminum. Principium quidem ad apprehensionem pertinet, terminus autem ad iudicium; ibi enim cognitio perficitur."

[9] Ibid. "Principium igitur cuiuslibet nostrae cognitionis est in sensu, quia ex apprehensione sensus oritur apprehensio phantasiae, quae est motus a sensu factus, ut dicit philosophus, a qua iterum oritur apprehensio intellectiva in nobis, cum phantasmata sint intellectivae animae ut obiecta, ut patet in III de anima." Note the use of the term *apprehension* across the types of knowing.

apprehension is not propositional, rather, we intentionally *receive the form* of the object known at each level of knowledge. At this level of thought the human mind is not making anything new—its understanding *consists in* its conformity to the object. It is only when we *make claims* that we begin to produce something which does not conform to the external reality (namely, a proposition). Reality itself is not composed along subject-copula-predicate lines. As Aquinas says elsewhere.

> Just as the true is found more in the intellect than in things, so also is it found more in the intellect's act of composition and division than in its act of forming the *quiddities* of things. For the nature of the true consists in the conformity of the intellect to a thing. Now, nothing conforms to itself, rather, only *different things* can be conformed. So, the nature of truth is first found in the intellect when the intellect begins to have something proper to itself (something the thing outside the soul does not have) albeit corresponding to that thing, such that a conformity may be discovered there. When the intellect forms the quiddities of things, it merely possesses a likeness of a thing existing outside the soul, just as sensation possesses a likeness when it receives the form of a sensible thing. But when the intellect begins to judge about what it has apprehended, then its judgment is something proper to it—something not found in the thing—which judgment is said to be *true* when it conforms to what exists outside the mind.[10]

Aquinas rightly calls the *beginning* of all our knowledge *apprehension*, then, since at this point our knowledge consists simply in a reception of what is there. Notably, however, the reception of sensation is first, followed by the reception proper to the imagination, and, finally, the reception of the *quiddity* of things by the intellect itself—a *tri-partite* beginning, as it were.

[10] *De Veritate*, 1 3 *corpus.* "Dicendum, quod sicut verum per prius invenitur in intellectu quam in rebus, ita etiam per prius invenitur in actu intellectus componentis et dividentis quam in actu intellectus quidditatem rerum formantis. Veri enim ratio consistit in adaequatione rei et intellectus; idem autem non adaequatur sibi ipsi, sed aequalitas diversorum est; unde ibi primo invenitur ratio veritatis in intellectu ubi primo intellectus incipit aliquid proprium habere quod res extra animam non habet, sed aliquid ei correspondens, inter quae adaequatio attendi potest. Intellectus autem formans quidditatem rerum, non habet nisi similitudinem rei existentis extra animam, sicut et sensus in quantum accipit speciem sensibilis; sed quando incipit iudicare de re apprehensa, tunc ipsum iudicium intellectus est quoddam proprium ei, quod non invenitur extra in re. Sed quando adaequatur ei quod est extra in re, dicitur iudicium verum." Emphases mine.

As to how knowledge *terminates*, Aquinas once again makes a threefold distinction:

> Now knowledge terminates in different ways, sometimes in the senses, sometimes in the imagination, and sometimes in the intellect alone. For sometimes properties and accidents revealed by the senses adequately express a thing's nature, in which case the intellect's judgment about it must conform to what the senses reveal. All natural things, determined by sensible matter, are of this kind. Therefore, our knowledge in *natural science* must terminate in sensation, such that we make judgments about natural things according to what the senses show us, as is clear from [Aristotle's] *De caelo et mundo*. Whoever neglects the senses when thinking about natural things, then, will fall into error.[11]

Aquinas argues here that the strength of the empirical sciences consists in their conformity to fact. No hypothesis, no matter how elegant or intuitively attractive to us, can be sustained in the face of contrary findings. A consistently formulated system or an *on paper* solution will always be inadequate when it comes to thinking about natural things, whose *court of last resort* is always what we see, touch, taste, hear, and smell.

Is the same true of the other speculative sciences? Recall that the question Aquinas is asking in this article is whether recourse to the imagination is helpful, or harmful, when thinking about divine matters. One might ask the same question about the senses, if it were not obvious that knowledge in immaterial realities cannot terminate on the basis of what the senses reveal. "God does not exist since no one has ever seen Him" is a method-error, as it turns out, since it does not respect the *way* in which one must go about investigating immaterial realities.

[11] *In De Trinitate*, 6 2 *corpus*. "Sed terminus cognitionis non semper est uniformiter: quandoque enim est in sensu, quandoque in imaginatione, quandoque autem in solo intellectu. Quandoque enim proprietates et accidentia rei, quae sensu demonstrantur, sufficienter exprimunt naturam rei, et tunc oportet quod iudicium de rei natura quod facit intellectus conformetur his quae sensus de re demonstrat. Et huiusmodi sunt omnes res naturales, quae sunt determinatae ad materiam sensibilem, et ideo in scientia naturali terminari debet cognitio ad sensum, ut scilicet hoc modo iudicemus de rebus naturalibus, secundum quod sensus eas demonstrat, ut patet in III caeli et mundi; et qui sensum neglegit in naturalibus, incidit in errorem."

What About Mathematics?

> Our judgment about other things does not depend upon what the senses perceive, because, though [these other things] exist in sensible matter, their defining *ratios* are abstracted from it, and we judge of a thing first and foremost according to its defining *ratio*. But because their defining *ratios* are not abstracted from *every* kind of matter but only sensible matter, and because something imaginable remains after its sensible conditions have been set aside, we must judge about such things according to what our imagination shows us. The mathematicals are this kind of thing. Therefore, the mathematical knowledge we have by way of judgment must terminate in the imagination and not in the senses, because mathematical judgment goes beyond the apprehension of sensation. We sometimes do not reach the same judgment about the mathematical line and the sensible line. For example, that a straight line touches a sphere at only one point is true of the abstracted straight line but not of a straight line in matter, as is said in [Aristotle's] *De Anima*.[12]

Recalling that one of the problems with which we came away from our first review of Aristotle's mathematical philosophy was the *precision-problem*, Aquinas' insistence upon resolving mathematical judgments, not to what the senses tell us, but to what the imagination reveals, makes much sense. After all, it is simply false to say that the angles of a triangle drawn on paper—howsoever precisely—are equal to one-hundred and eighty degrees. But neither is it the imagination that *makes* the judgment, nor was this true for the natural philosopher (nor, as it turns out, for the metaphysician). For judgment is an act of the intellect, which begins with, and is based upon, what the senses, imagination, and intellect apprehend, but which goes beyond those apprehensions. We do not know that a

[12] Ibid. "Quaedam vero sunt, quorum iudicium non dependet ex his quae sensu percipiuntur, quia quamvis secundum esse sint in materia sensibili, tamen secundum rationem diffinitivam sunt a materia sensibili abstracta. Iudicium autem de unaquaque re potissime fit secundum eius diffinitivam rationem. Sed quia secundum rationem diffinitivam non abstrahunt a qualibet materia, sed solum a sensibili et remotis sensibilibus condicionibus remanet aliquid imaginabile, ideo in talibus oportet quod iudicium sumatur secundum id quod imaginatio demonstrat. Huiusmodi autem sunt mathematica. Et ideo in mathematicis oportet cognitionem secundum iudicium terminari ad imaginationem, non ad sensum, quia iudicium mathematicum superat apprehensionem sensus. Unde non est idem iudicium quandoque de linea mathematica quod est de linea sensibili, sicut in hoc quod recta linea tangit sphaeram solum secundum punctum, quod convenit rectae lineae separatae, non autem rectae lineae in materia, ut dicitur in I de anima."

straight line touches a circle in a single point but imagining to be so, rather, the straight line we construct touching the circle is *shown to be* tangent, by a demonstration. What is it that the imagination reveals that the senses do not? The truth lying in the construction. Most precisely, Aquinas says in the above text that *mathematical judgment goes beyond the apprehension of sensation*, literally *iudicium mathematicum superat apprehensionem sensus.*

In one sense it is odd for Aquinas to say this in this case, since that is true for *all* judgments as over against the apprehensions with which knowledge begins, as he said above. Even judgments made by the natural philosopher go beyond the apprehension of the senses. What he means, rather, is that mathematical judgments surpass what the senses alone can *reveal to us*, which is why they are said to terminate in the imagination. It is on the basis of what we apprehend through imaginative constructions that we justify mathematical conclusions. This is clear from how Aquinas concludes this article:

> We are therefore able to use the senses and imagination as *principles* in our knowledge of divine things, but not as *terms*—in the sense that *we would judge them to be the sort of objects sensation or the imagination apprehend*. Now to go to something is to terminate in it. Therefore, we should go neither to the imagination nor to the senses in divine science, to the imagination but not to the senses in the mathematical sciences, and to the senses in the natural sciences.[13]

At this point, we have seen some sense in Aquinas' insistence upon the imagination as the distinctive *locus* for resolving our mathematical questions, through demonstrated conclusions we come to know on the basis of what it reveals about the nature of the mathematicals. Now, Aquinas has just said that the imagination operates as a beginning—the second phase of the tripartite apprehension upon which science is ultimately based. This, too, makes sense when we consider what in us makes *episteme* possible.

[13] Ibid. "Uti ergo possumus in divinis et sensu et imaginatione sicut principiis nostrae considerationis, sed non sicut terminis, ut scilicet iudicemus talia esse divina, qualia sunt quae sensus vel imaginatio apprehendit. Deduci autem ad aliquid est ad illud terminari. Et ideo in divinis neque ad imaginationem neque ad sensum debemus deduci, in mathematicis autem ad imaginationem et non ad sensum, in naturalibus autem etiam ad sensum. Et propter hoc peccant qui uniformiter in his tribus speculativae partibus procedere nituntur."

Aristotelian Hints

As Aristotle argues at the very beginning of his *Metaphysics*, we all naturally desire to know, as evidenced by the progression we discover among animals, all of whom sense, but only some of whom retain images of what they have sensed, namely, memories. From many memories comes experience and, from experience, art and science in us.[14]

Aquinas elaborates, showing the need for a connection between the senses, the imagination, and memory.

> Some animals have sensation but not memory, which arises from sensation. For memory follows upon imagination, which is a movement caused by the senses in their act of sensing, as [Aristotle says in] Book II of De anima. But in some animals sensation does not give rise to imagination, so neither do they have memory. Imperfect animals like shellfish, which do not move themselves about, are of this kind. For since sense knowledge enables animals to provide for life's necessities and for their characteristic operations, then animals which make progress toward something distant by means of local motion must have memory. For if the preconceived goal which moves them did not remain in them through memory, they could not make progress toward the intended end. But, in order to do what is characteristic to them, it is enough that immobile animals receive a sense quality [immediately] present to them, since they do not move toward anything distant [anyway]. So it is that such animals have an indefinite movement due solely to a confused imagination, as [Aristotle] points out in Book III of De anima.[15]

Crucial to Aristotle's (and Aquinas') overall argument is the capacity to retain sensed forms. The imagination alone is insufficient to allow for

[14] *Metaphysica*, I 1.

[15] *In Metaphysicorum*, I 1. "Quaedam enim sunt, quae licet sensum habeant, non tamen habent memoriam, quae ex sensu fit. Memoria enim sequitur phantasiam, quae est motus factus a sensu secundum actum, ut habetur in secundo de anima. In quibusdam vero animalibus ex sensu non fit phantasia, et sic in eis non potest esse memoria: et huiusmodi sunt animalia imperfecta, quae sunt immobilia secundum locum, ut conchilia. Cum enim animalibus cognitio sensitiva sit provisiva ad vitae necessitatem et ad propriam operationem, animalia illa memoriam habere debent, quae moventur ad distans motu progressivo: nisi enim apud ea remaneret per memoriam intentio praeconcepta, ex qua ad motum inducuntur, motum continuare non possent quousque finem intentum consequerentur. Animalibus vero immobilibus sufficit ad proprias operationes, praesentis sensibilis acceptio, cum ad distans non moveantur; et ideo sola imaginatione confusa habent aliquem motum indeterminatum, ut dicitur tertio de anima."

intentional movement, to be sure, but, more importantly, it is by means of a number of memories—a number of retained forms—that experience, and therefore art and science, is possible. Thus Aquinas, following Aristotle: Just as one experiential knowing arises out of many memories of a thing, so does a universal notion about all similar things arise out of the apprehension of many experiences.[16] If, as these men insist, all human knowledge originates in the senses, then apprehending the *commonality* among many like things requires that they be present to our minds all at once. But successive instances of sensing like things can only be present at once if some or all are retained for future *inspection*.

This is the need for the imagination at the *beginning* of all human cognition—the second phase of the threefold apprehension Aquinas mentioned above. Universality itself pre-requires imagination, and *episteme*, of course, is all about universality, mathematics not the least. Imagination as *retentive*, then, is not what distinguishes mathematical judgments from other sorts, since that is part of the method of *every* science—indeed, of all human knowledge.

The Representative and Creative Imagination

Of course, imagination has another sense, or function, beyond representing the forms of sensations past to the view of the mind. In fact, in our day, imagination as *creative* or *productive* comes to mind much more readily, so much so that one might call into question whether the imagination is any part of simply *receiving what is there*. The distinction between the two functions is not lost on Aquinas.

> As to the memorative power, we have not only memory—as do other animals when they recollect the past all at once—but also *recollection*, where we syllogistically (as it were) search for a memory of the past among the individual intentions. For his part Avicenna posits a fifth power between the estimative and imaginative powers, whereby we combine and separate imagined forms, as when from the imagined form of gold and the imagined form of a mountain we produce the one form of a golden mountain, which we have never seen. But this operation is not found in animals other than man, in whom the imaginative power suffices for this purpose. Averroes also attri-

[16] Ibid. "Nam sicut ex multis memoriis fit una experimentalis scientia, ita ex multis experimentis apprehensis fit universalis acceptio de omnibus similibus."

> butes this action to us in his book *On sensation and the sensibles.* So, we need posit no more than four interior powers of the sensitive part—the common sense, the imagination, and the estimative and memorative powers.[17]

Unlike the Islamic philosopher Averroes, Aquinas explains how we can creatively imagine by referencing human reason. Just as memory in us also takes the form of recollection—the rational searching out of a specific memory—so, too, our imagination can exhibit aspects of thought from its conjunction with a mind in us. In short, we can imaginatively create because we can imagine *and* because we are rational. Add to this that we are also *free*—we are the principle of our own actions—and we can explain why our imagination is not *limited* to what we have directly sensed, rather, we can combine or separate images as we please.

This connection to reason is more explicit in this passage from *De veritate*:

> Some of our knowing powers know things only by *receiving* and not by making something out of what they have received. For example, sensation simply apprehends the form it has received, nothing more. Others not only apprehend what they have received but are also able to make something out of it, as is clear with the imagination, which forms the image of a golden mountain from the received images of gold and of a mountain, as does the intellect when it formulates the *quiddity* of a species from the forms of its genus and difference.[18]

[17] *Summa Theologiae*, I 78 4 *corpus.* "Ex parte autem memorativae, non solum habet memoriam, sicut cetera animalia, in subita recordatione praeteritorum; sed etiam reminiscentiam, quasi syllogistice inquirendo praeteritorum memoriam, secundum individuales intentiones. Avicenna vero ponit quintam potentiam, mediam inter aestimativam et imaginativam, quae componit et dividit formas imaginatas; ut patet cum ex forma imaginata auri et forma imaginata montis componimus unam formam montis aurei, quem nunquam vidimus. Sed ista operatio non apparet in aliis animalibus ab homine, in quo ad hoc sufficit virtus imaginativa. Cui etiam hanc actionem attribuit Averroes, in libro quodam quem fecit de sensu et sensibilibus. Et sic non est necesse ponere nisi quatuor vires interiores sensitivae partis, scilicet sensum communem et imaginationem, aestimativam et memorativam." Emphasis mine.

[18] *De veritate*, 8 5 *corpus.* "Est enim aliqua cognitiva potentia quae cognoscit tantum recipiendo, non autem aliquid ex receptis formando; sicut sensus simpliciter cognoscit illud cuius speciem recipit, et nihil aliud. Aliqua vero potentia est quae non solum cognoscit, secundum quod recipit; sed etiam ex his quae recipit, potest aliquam aliam speciem formare; sicut patet in imaginatione, quae ex forma auri recepta et forma montis, format quoddam phantasma aurei montis. Et similiter est in intellectu, quia ex forma generis et differentiae comprehensis format quidditatem speciei." Emphasis mine.

By virtue of their conjunction with our minds, memory and imagination—though not themselves capable of making judgments—do participate in our intellectual inquiries, as evidenced by our recollections and creative imaginings. It is the latter Aristotle references when he says: "[Imagination] is in us whenever we wish (for it is something constructed before our eyes, like those who make and place images in mnemonics),"[19] but also "…sensations are always true, whereas many of the imaginings we make are false."[20] As Aquinas says in his *Commentary*:

> …when actually sensing our senses are always true, for they are not deceived about their proper sensibles, whereas imaginings are usually false, for there is usually nothing [in reality] that corresponds to the imagination.[21]

Imagination and the Truth

This brings us to this chapter's main difficulty. If, as Aquinas insists, mathematical judgments require constructive demonstrations and terminate in the imagination, but the imagination often presents falsehoods, what distinguishes constructed falsehoods from constructed truths? What is to say that mathematics does *not* consist of (at least some) *useful fictions*, in (fictionalist) Hartry Fields' own words? If imagined fictions yield results as much as imagined realities, why bother with distinguishing them at all? Moreover, why limit mathematical imaginings at all—especially since the imagination is naturally free and falsity seems not to be the issue?

We noted earlier that some modern philosophers—even Thomistic philosophers—are optimistic when it comes to Aquinas and modern mathematical developments. Thomas Anderson, for example:

> As for the freedom of the mathematician in his demonstrations of existence, it would seem that he is at liberty to construct in imagination *any*

[19] *De Anima*, III 3 427b18–19. τοῦτο μὲν γὰρ τὸ πάθος ἐφ' ἡμῖν ἐστιν, ὅταν βουλώμεθα (πρὸ ὀμμάτων γὰρ ἔστι τι ποιήσασθαι, ὥσπερ οἱ ἐν τοῖς μνημονικοῖς τιθέμενοι καὶ εἰδωλοποιοῦντες).

[20] Ibid., 428a11–12. εἶτα αἱ μὲν ἀληθεῖς ἀεί, αἱ δὲ φαντασίαι γίνονται αἱ πλείους ψευδεῖς.

[21] *In De Anima,* 3 5 645. "Sensus secundum actum semper sunt veri; non enim sensus decipitur circa proprium sensibile: sed phantasiae ut plurimum sunt falsae. Non enim respondet imaginationi ut plurimum."

> *mathematicals he can*, and this would apparently mean any quantities whose existence is not self-contradictory.[22]

That both Aristotle and Aquinas see mathematicians (here, geometers) as dealing with individual straight lines touching circles at one point—something which everyone denies about physical lines and circles—leaves us with one conclusion: such things are constructed by the human imagination, which is not only representative but creative, which is something on which the two philosophers also agree. The difficulty is that our created imaginings (Aristotle's αἱ φαντασίαι γίνονται and Aquinas' *ex his quae recipit, [imaginatio] potest aliquam aliam speciem formare*) are usually false.[23] We have already shown examples of constructed mathematicals both men would exclude from mathematics proper, as being incompatible with continuous quantity in three dimensions or the arithmetical unit.

Aquinas' creative imagination does imply freedom in mathematical constructions. So does Aristotle's, in fact. Anderson is *correct* in saying that, barring self-contradiction, mathematicians are at liberty to construct as they wish. Still, it does not follow that such constructs yield objectively true judgments, as being virtually contained in real mathematical principles abstracted from real physical things. The creative imagination is intrinsically subject to error—necessary to mathematics though it may be—in that what we imaginatively construct from units and lines may have no more of *episteme* to it than mountains of gold. For purposes of *episteme*, as we have argued in Chap. 6, our imaginative constructions must not only arise from real principles apprehended through the senses, they must be shown to be logically compatible with those principles. One can be led to think that *some eventual unit length* could be found which measures both the side of a square and its diagonal—progressively imagining a smaller and a smaller still is virtually effortless. It requires reason to see the contradiction such an assertion entails. The connection to the real which *episteme* requires comes at its own cost. The vast mathematical worlds we merely imagine have a undeniable beauty of their own. Even so, it is not the beauty of the real.

[22] Anderson, "Aristotle and Aquinas on the Freedom of the Mathematician," 252. Emphasis mine.

[23] For example, it would be unwise to take what the imagination requires regarding the limits of the physical universe (we cannot *imagine* its outer limit without imagining some space beyond) as demonstrating that fact. Cf. Lucretius, *On the Nature of Things* (Indianapolis: Hackett Publishing Company, 2001), I, 969–84.

In the next part of this book we will review some modern and recent developments in mathematics, with a view to learning what problems they have encountered historically and whether Aquinas' mathematical philosophy can explain why these problems have occurred and how they might have been avoided.

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PART III

Modern Philosophies of Mathematics

CHAPTER 10

Going Beyond

Numerical Extensions

We have already addressed numerical extensions and the response given them by Aristotle and Thomas Aquinas.[1] While not a modern development—by any means—a general acceptance of such extensions is. Take zero and the negative numbers. Among the complications found in zero is that it has varied senses—from *the additive identity* to *a position on the number line* to *false*. Bertrand Russell notes in his *Introduction to Mathematical Philosophy* that there are nonidentical zeroes.[2] In some applications it represents a possible result (the empty set) and in others an exception—for example, rationals are closed under division except for 0. Zero and the negative integers are both included among the numbers closed to subtraction.[3] Then, apart from division by zero, the rationals are closed to division *by definition*—alternatively, the *free use of division* upon the integers extends that group of numbers to the rationals.

[1] See Chap. 6, above.

[2] Cf. Bertrand Russell, *Introduction to Mathematical Philosophy* (London: George Allen & Unwin, 1919), 63.

[3] See our use of the closure property for subtraction (or *the free application of subtraction)* as a principle for extending the whole numbers, above.

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As to the irrationals, among the more famous are √2 and π, first discovered in a geometrical context.[4] Since irrationals are *incommensurable*—there is no unit whereby the numerator and denominator can be measured—it is easy to recall why Aristotle and Thomas Aquinas excluded them from the *episteme* of mathematics.

The imaginary numbers are another instance of an *extension* consequent upon the *free application* of basic numerical operations on numbers: closing numbers to subtraction yields the integers and closing numbers to division yields the rationals (except for zero). Moreover, as we noted in the previous section, closing the positive rationals to the extraction of roots yields the irrationals, or, alternatively, the *free application* of the arithmetical operations upon *positive ratios of any sort* yields the reals. In the case of the imaginary numbers, we close the numbers to the extraction of roots, yielding the complex numbers.[5] Since root-values are factors, finding roots is a special case of factorization, which is a sort of division (just as exponentiation is a kind of multiplication).

Even so, say Aristotle and Thomas Aquinas, to the extent that such numbers compromise the very nature of mathematical unity (the indivisible measure of plurality), they cannot be legitimized under the strict notion of mathematics they have developed.

Geometrical Extensions

What of historical developments regarding the mathematics of dimensional space—geometry? We will address two, topology and non-euclidean geometries, in the remainder of this chapter. We will then end our consideration of modern and recent views with a separate chapter on Cantor's transfinites and the famous controversy it spawned.

As to the mathematics of space, topology concerns the unchanging relational characteristics among changeable spatial structures—so-called *rubber sheet* geometry. Historically, it took its rise from a famous problem presented by Euler in his *Solutio problematis ad geometriam situs pertinentis*, namely, the *Königsberg bridge problem*. At issue was the question whether one could journey across each of seven bridges only once.

[4] √2 is the numerical expression of the ratio between a square's diagonal and its side and π is the numerical expression of the ratio between a circle's circumference and its diameter.

[5] For example, the root of $i^2 + 1 = 0$ is √-1. The complex numbers are thereby said to be *algebraically closed*.

Importantly, the *actual* position of the crossings (the vertices of a polyhedron in standard geometry) was irrelevant to the solution of the problem—hence a map of the bridges could be re-arranged in multiple ways and the problem would be unchanged. Euler proved that such a crossing is impossible—a conclusion regarding a new aspect of three-dimensional space, namely, the *relationships* obtaining between edges and vertices in such spaces. Other famous examples of topological structures include the mobius strip and the four-color map. Such structures are included among others in clusters, such as vector-edge graphs in graph theory and discrete structures generally in discrete mathematics.

Where might Aristotle and Thomas Aquinas stand on such extensions? On the one hand, they exhibit some of the same features one would find among more traditional constructs in geometry: they are abstract, universal, and necessary (e.g., *abstracting from* the static shapes formed by seven bridges and the paths we take to them, one cannot trace a *unique* path). On the other hand, mathematics, for Aristotle and Aquinas, seems to be exactly *about* what is unchanging in three-dimensionality and its indivisible unit. In their terms, what would be the *nature* under consideration in the seven-bridges problem?[6]

We noted above that Aristotle divides reality into ten *categories*, among which he includes the category of *relation*.

> What is said to be *essentially* signifies just as the figures of the categories do: in howsoever many ways these are said, in so many ways does being signify. Now since, of these categories, some signify what it is, others of what sort, others how much, others in relation to what, others what it does or suffers, others where, and others when, to each of these there [corresponds] a being which it signifies.[7]

Earlier, we used this passage as the foundation of Aristotle's account of the *episteme* of quantity; after all, *episteme* is knowledge of the real. If continuous quantity in three dimensions and the indivisible measure of

[6] *Episteme* requires that one know that the principles of demonstrating are true, the nature and existence of the subject of demonstration, and the nominal definition of the property being demonstrated to inhere in that subject. Cf. *Analytica Posteriora*, I 1 71a11–16.

[7] Metaphysica, V 7 1017a23–28. καθ αὐτὰ δὲ εἶναι λέγεται ὅσαπερ σημαίνει τὰ σχήματα τῆς κατηγορίας· ὁσαχῶς γὰρ λέγεται, τοσαυταχῶς τὸ εἶναι σημαίνει. ἐπεὶ οὖν τῶν κατηγορουμένων τὰ μὲν τί ἐστι σημαίνει, τὰ δὲ ποιόν, τὰ δὲ ποσόν, τὰ δὲ πρός τι, τὰ δὲ ποιεῖν ἢ πάσχειν, τὰ δὲ πού, τὰ δὲ ποτέ, ἑκάστῳ τούτων τὸ εἶναι ταὐτὸ σημαίνει.

plurality are not real, neither is there the science of mathematics. As to the other categories, Aristotle has this to say of relation—in particular of relations that pre-suppose *quantity*:

> [Relations] of the first sort are numerical—either simply, or determinately—whether in relation to the unit or to numbers, e.g., *the double* is determinately [related to] the unit but *the very much more* is numerically, but not determinately, related to the unit (as being in 'this relation or 'that one').[8]

Aristotle goes on to specify the kinds of relations possible (both determinate and indeterminate) for quantities of various sorts. Even for indeterminate relations, however, the notion of *measure* and *measured* is preserved:

> What contains is in a numerically indeterminate relation to what is contained. Number is commensurate. Number is not said of what is not commensurate, whereas what contains is, in relation to what is contained, so much and more—which is indeterminate. For it might happen to be either equal or unequal [to what is contained].[9]

All numerical relations are either equalities or inequalities and, among inequalities, are either determinately or indeterminately unequal.[10] But not all quantitative relations are numerical: indeed, what seems to distinguish the mathematical sciences from all others is that they remain within the realm of the equal and unequal: "What is most proper to quantity is that it is said to be equal and unequal."[11] In elaborating upon what Aristotle is saying in the passages from the *Metaphysics*, Aquinas explains how a consideration of relations (or proportions) does, and does not, apply to the sciences of mathematics, underscoring once again the fundamental character of the unit of measure for the science of arithmetic.

[8] Ibid., V 15 1020b32–1021a14. λέγεται δὲ τὰ μὲν πρῶτα κατ' ἀριθμὸν ἢ ἁπλῶς ἢ ὡρισμένως, πρὸς αὐτοὺς ἢ πρὸς ἕν (οἷον τὸ μὲν διπλάσιον πρὸς ἓν ἀριθμὸς ὡρισμένος, τὸ δὲ πολλαπλάσιον κατ' ἀριθμὸν πρὸς ἕν, οὐχ ὡρισμένον δέ, οἷον τόνδε ἢ τόνδε.

[9] Ibid., 1021a4–8. τὸ δ' ὑπερέχον πρὸς τὸ ὑπερεχόμενον ὅλως ἀόριστον κατ' ἀριθμόν: ὁ γὰρ ἀριθμὸς σύμμετρος, κατὰ μὴ συμμέτρου δὲ ἀριθμὸς οὐ λέγεται, τὸ δὲ ὑπερέχον πρὸς τὸ ὑπερεχόμενον τοσοῦτόν τέ ἐστι καὶ ἔτι, τοῦτο δ' ἀόριστον: ὁπότερον γὰρ ἔτυχέν ἐστιν, ἢ ἴσον ἢ οὐκ ἴσον).

[10] For an extended discussion of determinate and indeterminate numerical proportions, see *In Metaphysicorum* V 17 1006–1021.

[11] *Categoriae*, 6 6a26–27. ἴδιον δὲ μάλιστα τοῦ ποσοῦ τὸ ἴσον τε καὶ ἄνισον λέγεσθαι.

> When one quantity is said to be *greater than* another, or is related to it *as what contains to what is contained*, not only is this *ratio* not taken according to some determinate species of number, it is not even taken according to number [at all], since every number is commensurable with every other. For all the numbers have a common measure, namely the unit, whereas what contains and what is contained are not taken according to some numerical commensuration. That is because the relation between what contains and what is contained consists in this, that it be *so much and something more*, which is indeterminate, whether it be commensurable or not. Whatever quantity be considered, it is either equal or unequal (to another). If unequal, one will be *so much and something more*, even if [the more] is not commensurable.[12]

According to Aristotle and Thomas Aquinas, then, relations need not be quantitative and quantitative relations need not be numerical. What distinguishes the mathematical sciences among others is that they are properly concerned with quantity and, more specifically, with the relations which obtain among quantities of various sorts.

The reality of quantitative relations (e.g., the relation of equality among the sides of the polygon first constructed in Euclid's geometry, the equilateral triangle) is a necessary condition for the *episteme* of mathematics. If topology is concerned with spatial relations other than those subsumed under equality and inequality, could we not include the study of such relations in a discipline of its own? Are they not among the properties necessarily following from the assumption of three-dimensional space?

A simple example is *orientability*. When Euclidean surfaces are represented in three-dimensional space, a natural left and right (called *handedness*) can be discerned. So too, pathwise, natural clockwise and counter-clockwise directions. Such surfaces are said to be *oriented*.[13] It is a legitimate question to ask whether all real-world surfaces have this property.[14] But it is not a question asked in standard Euclidean geometry, which (as we have seen) is concerned with the general relations of equality and inequality. If *orientation* (or *non-orientation*, for that matter) is a real

[12] *In Metaphysicorum*, V 17 1021.

[13] The term derives from the Latin for 'east.' Definitions of and tests for orientability vary greatly, from the very abstract to the immediately practical (e.g., moving a nonsymmetrical object along the surface and discerning whether it presents its mirror image upon reaching the origin).

[14] Of course, the answer is 'no,' as instanced by the 1858 discovery by Listing and Möbius of a non-orientable surface—commonly known as a *Möbius strip*.

property belonging to real space, then a study of it and the quantities to which it belongs can rise to the level of *episteme*.

Interestingly enough, Thomas Aquinas supplies an account of orientation-relations and even—ideal for our purposes—addresses the question whether such relations are real. At issue is the question whether *handedness* or *orientability* are real properties of real spaces.

Aristotle's account of relations contains an important distinction, since some correlatives are mental and not real.

> It does not seem true in every instance that correlatives come about at the same time: for the thing known is prior to the knowing of it. We usually acquire knowledge of things that *already* exist.[15]

Whereas Aristotle's own examples focus upon such one-way relations as knowing and perceiving, Thomas Aquinas supplies an additional example of relations that obtain in the orientation of spaces.

> Some relations are not something really inhering in that of which they are predicated. This sometimes happens on the part of both terms, as when we say that the self is the same as itself: for this relation of identity could be multiplied *in infinitum* if anything were the same as itself by means of an additional relation: For, clearly, any thing is the same as itself. Therefore, this is a relation only with respect to reason, insofar as reason takes one and the same thing as being the two terms of the relation. And this happens in many other cases. … So, too, when a column is said to be to the right of an animal: for right and left are real relations in animals, since the determinate powers upon which such relations are based are found in them. But [right and left] in the column are not real relations but only with respect to reason, because [columns] do not have the determinate powers upon which such relations are based.[16]

[15] *Categoriae*, 7 7b22–25. οὐκ ἐπὶ πάντων δὲ τῶν πρός τι ἀληθὲς δοκεῖ τὸ ἅμα τῇ φύσει εἶναι· τὸ γὰρ ἐπιστητὸν τῆς ἐπιστήμες πρότερον ἂν δόξειεν εἶναι. ὡς γὰρ ἐπὶ τὸ πολὺ προϋπάρχόντων τῶν πραγμάτων τὰς ἐπιστήμας λαμβανομεν.

[16] *In Physicorum* 5 3 667. Sunt enim quaedam relationes quae non sunt aliquid realiter in eo de quo praedicantur. Quod quidem quandoque contingit ex parte utriusque extremi, sicut cum dicitur idem eidem idem: haec enim identitatis relatio in infinitum multiplicaretur, si quaelibet res esset sibi eadem per relationem additam: manifestum est enim quod quodlibet sibi ipsi est idem. Est ergo haec relatio secundum rationem tantum, inquantum scilicet unam et eandem rem ratio accipit ut duo extrema relationis. Et similiter est in multis aliis. … Et similiter est cum columna dicitur dextra animali: dextrum enim et sinistrum sunt relationes reales in animali, quia in eis inveniuntur determinatae virtutes, in quibus huiusmodi relationes fundantur: in columna autem non sunt secundum rem, sed secundum rationem tantum, quia non habet praedictas virtutes, quae sunt fundamenta harum relationum.

Merely mental relations are understood only by reference to something fundamentally real—in this case, by analogy to things *really* having a right side in virtue of *really* having a front. Were there a special study addressing the relations of right and left—*orientation*—or the rotational directions clockwise and counter-clockwise[17], we can see both Aristotle and Aquinas counting it among the sciences, properly speaking.

On the other hand, given that—for Aristotle and Aquinas both—some relations *are* real, why is it that there is no separate *episteme* having relation as its ultimate subject?

Aristotle and Aquinas see relation as an accident inhering in substances, true, though not *immediately*; that is, relations themselves have their foundation in accidents of a prior sort, which are, in turn, founded in substances.

Taking Aristotle's own example of knowledge, our capacity to know is really related to its objects, which is to say it is the sort of potency which can be actualized only in a determinate way.[18] On the other hand, since it is *passive*, the mind's coming to know is not accompanied by a like change in the object known. Being *really able to know* is the foundation of the one-way relation of the knower to the known. The *determination* of that ability *to* a single kind of actuality—the relation, in other words—cannot be separated from the kind of thing which has that determination. Being a real *knower* presupposes that one is something of another sort: capable. If one were to engage in a systematic consideration of that specific relation, one could only do so in the light of a consideration of living capacities, specifically, a mind. A separate account of this relation—an *episteme* of the knower, in other words—cannot be abstracted from an inquiry into living things. To consider *knowers* as such, then, is to consider knowing beings—a subset of *natural philosophy*.

What about other relations? We have discussed equality and inequality, for example, as being the proper concern of the mathematical sciences. If what we have just said is correct, however, there could (likewise) be no

[17] The *terms* 'clockwise' and 'counter-clockwise,' of course, reference the nonessential and artificial, much like the arbitrariness which characterizes the number of members in a non-natural whole (cows in the field, for example). Even so, there are natural and objective instances of linear and rotational *directedness*. Time is one of the most fundamental, though more mundane examples of natural faces and the rotation of our own planet suffice to make the point.

[18] We spoke of determinate potencies above when we considered Aristotle's and Aquinas' account of knowing things virtually.

separate *episteme* of equality and inequality—equality and inequality are relations obtaining between or among *quantities*, in which case we could not separate their consideration from that of their natural foundation in the real. No surprise, then, that the proper concern of the mathematician includes those relations which accrue to quantity, as such.

But *are* there relations belonging to the mathematical unit and continuous quantity in three dimensions[19] beyond equality and inequality?

It finally comes down to this: does topology constitute an additional—a new—consideration of the nature of space-relations? Are *orientation, handedness*, and *rotational direction*, for example, real and objective relations which accrue to continuous quantity in three dimensions, abstractly and as such?

On the one hand, the field of topology seems so to regard them. In pronouncing a Möbius strip non-orientable and a Euclidean surface orientable, the topologist is not open to qualifying these claims in terms of perspective, as if one were to say that a Euclidean surface can be regarded as non-orientable *from a certain point of view* and the Möbius strip likewise. In fact, it is their objective *difference* that so struck those who originally discovered non-orientability. Much as the Pythagoreans with their famous theorem and Euclid with the five (and only five) regular solids, it is the *objectively marvelous* (so to speak) in the Möbius strip that drives the scientific mind to find the real difference here—and this gives rise to topology itself.

On the other hand, it is necessary to ask whether the objectivity of the *orientable*, *handed*, and *rotationally directed* relations is sufficient to include them as an extension of *mathematics*. Even Euler (at least initially) was not certain such problems were mathematical ones. Upon receiving a request to solve the seven-bridges problem, he responded:

> Thus you see, most noble Sir, how this type of solution bears little relationship to mathematics, and I do not understand why you expect a mathematician to produce it, rather than anyone else, for the solution is based on reason alone, and its discovery does not depend upon any mathematical principle.[20]

[19] Recall that the mathematical unit and continuous quantity in three dimensions are the first subjects of the episteme of mathematics.

[20] As quoted in Brian Hopkins and Robin J. Wilson, "The Truth About Königsberg," *The College Mathematics Journal* 35 no. 3 (2004): 201.

At issue is the need for abstraction. We made the point above that a consideration of the relation of *knower* would not constitute a form of *episteme* different from that in which living beings having capacities to know are studied, namely, philosophy of nature. Arguably, a consideration of relations would pertain to that form of *episteme* whose proper concern is the foundation upon which such relations are based. This is true for the quantitative relations of equal and unequal, which are founded upon quantity, and, seemingly, for relations founded upon the categories of action and passion.[21]

What is the foundation of the relations of *orientation*, *handedness*, and *rotationally directed*? Can we understand such relations in terms of quantity alone, abstracted, as we say, from sensible matter, or is it necessary that we understand them in terms, not of continuous quantity in three dimensions, abstractly and as such, but as having a basis in some real capacity in a physical substance?

At least in one sense, we have already seen how Aquinas would regard *handedness*—where that means *having an objective right and having an objective left*. If *having a face* (*front*) is a precondition for right and left, then, however objective such relations might be, they are not properties of the three dimensional *continuum*, abstractly and as such—after all, mathematics abstracts from the particular sensible characteristics of its object, leaving only *substance subject to three dimensions*. If the properly topological sense of *handedness* abstracts from the sensible characteristics of real *continua*, then it, too, would be included in the *episteme* of mathematics, it, too, would be properly *geometrical*. On the other hand, if it necessarily presupposes *natural* aspects of the respective *continua* (like a real, as opposed to merely logical, *right* or *left*), it would properly belong to the consideration of the natural philosopher.[22]

[21] Both Aristotle and Thomas Aquinas maintain that all relations are founded upon quantity or action and passion. For example, see *Metaphysica*, V 15 and Thomas Aquinas, *Quaestiones Disputatae de Potentia* (Rome: Marietti Editori, Ltd., 1949), 1 7 9 *corpus*, hereafter *De Potentia*.

[22] Nor is this *placement* of such a study—in mathematics proper or in natural philosophy—of relatively little importance, overall. As Aristotle makes clear in many passages, not only do the various *epistemes* differ as to their formal objects, the methods whereby such knowledge is obtained differ accordingly. Final causality has no place in a geometric consideration of the five regular solids, though it would when considering solid shapes in chemistry or biology. Solid shapes, symmetry, and orientation are naturally *purposive* among living things, and so it is part of the natural philosopher's concern to study them. The only question is whether they are the concern of a special discipline which has *abstract* shape, orientation, and sym-

Non-Euclidean Geometries

One recent Aristotelian who has made much of this geometric extension—the inclusion of quantitative relations beyond equality and inequality in the discipline proper—is James Franklin, who defends a form of Aristotelian mathematical realism. We will leave off proposing our own answer to this immediate question to Chap. 13, where we plan to compare the Aristotelian and Thomistic account we are presenting here to other contemporary realist accounts—including Franklin's—showing where we might find compatibility and where we would depart from them.

Our second instance of a modern extension to the traditional account of three-dimensional space, taken abstractly and such, are the various historical forms of non-euclidean geometry. The account is well known. Among the postulates underpinning geometry, Euclid included an initial five:

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any centre and distance.
4. That all right angles are equal to one another.
5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.[23]

Postulate five became a matter of concern to mathematicians, even in Euclid's own day.[24] Following Aristotle's own account of *episteme*, *The Elements* begins with the first principles of geometry, those definitions, self-evident claims, and postulates, the understanding of which is presupposed to whatever conclusions are to be drawn.[25] Still—and from the beginning—mathematicians were not certain that postulate five was, in

metry as its proper objects, and about which it professes to determine what such things might be *in their essence*. Cf. Aristotle's discussion of the sphericity of the earth—the concern of the mathematician and the natural philosopher alike—in *Physica*, II 2.

[23] Euclid, *The Elements*, I Postulates.

[24] See Sir Thomas Heath's notes on Postulate 5 in Euclid, *The Elements*, beginning on page 202, as well as Roberto Bonola, *Non-Euclidean Geometry* (New York: Dover Publications, 1955), 1–21.

[25] Cf. Analytica Posteriora, I.2.72a7–24.

fact, such a first principle. There followed many attempts to demonstrate it from other geometric first principles, but to no avail. On the one hand, its complexity relative to the other postulates suggested that it was something less than a principle—where *principle* means incapable of analysis. On the other, though undeniably needed to establish Euclid's own conclusions, no such reduction to simpler principles was forthcoming. All in all, the fifth postulate seemed to be a different *kind* of first.

In 1829 or so, Russian mathematician Nikolai Lobachevsky published a treatise entitled "Geometrical Researches on the Theory of Parallels." Soon after, and independently, Hungarian mathematician János Bolyai published his "Science Absolute of Space." While specifically concerned with so-called *hyperbolic* geometries, the two are generally credited with establishing those areas of study collectively known as non-euclidean geometries. Both presented axiomatic systems which differed from Euclid's own principally through a denial of, or modification to, the parallel postulate. Famously, Bolyai remarked:

> It remains, finally (that the thing may be completed in every respect), to demonstrate the impossibility (apart from any supposition), of deciding *a priori*, whether Σ, or some S (and which one) exists.[26]

Bolyai was intent upon showing that Euclidean space—Σ, the space in which the parallel postulate holds true—is only one among conceivable spaces: hence, the *absolute* science of space, which transcends the specific kind of space under consideration. Bolyai designated that space "founded on the contrary hypothesis [contrary to the parallel postulate]" S. He goes on:

> All things which are not expressly said to be in Σ or in S, it is understood are enunciated absolutely, that is are asserted true whether Σ or S *is reality*.[27]

Bolyai was keenly aware that his extensions to Euclidean geometry could well be found only in the realm of the *purely* imaginative—that no real-world model of such spaces as S might be discovered. To be fair, it is not a *mathematical concern*.[28]

[26] János Bolyai, "Science Absolute of Space", in Bonola, *Non-Euclidean Geometry*, 48.
[27] Ibid., §15, p. 14. Emphasis mine.
[28] On the other hand, it is very much a concern for the philosopher of mathematics.

Much as with the topological extensions to traditional geometry, above, our own response to the problem of the *episteme* of non-euclidean geometries is mixed. There is no doubt that, in answer to Bolyai's question, if one could establish the objective reality of spaces beyond the three-dimensional one we naturally know there could be a systematic study of it in much the same way as we have now for traditional geometry. Reality, after all, is the first criterion for *episteme*—what is real we can know in this way.

On the other hand, it is unclear how we would *arrive* at our first notions of (say) four-dimensional space. Bolyai's Σ is naturally apprehended through sensation and serves to constrain[29] our imaginations to spatial length, width, and depth. Indeed, three-dimensionality *seems* inescapable—Immanuel Kant himself regards it as the *a priori* condition for *any* possible outer sensation, an *a priori form of sensible intuition*.[30] Were we to establish the reality of four-dimensional space, then, it could not be through direct sense-perception and imagination. Of course, the development of Albert Einstein's *general theory of relativity* in the early twentieth century presupposes—beyond the so-called *fourth dimension* of time—more than three *spatial dimensions*. What we *observe* as a curved path of light may be the straight passage of light through a curved space, for example. If we could establish that light propagates in an *undeviating* path through any space, then curved light would be evidence of curved space.

Of course, curved space—like anything whose existence can only be established indirectly—is *literally unimaginable*. Given the importance Aristotle and Thomas Aquinas give to the human imagination in mathematics—literal *constructability* as a solution to the precision-problem, for example—such an *episteme* of four (or more) dimensional spaces would not *be* geometry, though possibly analogous to it.

Much is made of the *modelings* of non-euclidean geometries to three-dimensional figures (e.g., the sphere or pseudo-sphere). Modeling geometry on a sphere, *great circles* become analogous to straight lines in standard three-dimensional space, for example, and *spherical triangles* have their combined angles greater than 180°. Importantly, however, such modelings supply us with, at best, an analogue to non-euclidean geometries.

[29] So to speak.

[30] Immanuel Kant, *Critique of Pure Reason*, trans. Werner Pluhar (Indianapolis: Hackett Publishing Company, 1996), Transcendental Doctrine of Elements, I, Transcendental Aesthetic, §1, no. 2, 26.

Spherical geometry proper—the geometry of drawing with curved lines on the surface of an (obviously) curved sphere—is still Euclidean. There is no contradiction in saying that triangles with curved sides have their combined angles greater than, or less than, triangles with straight sides. A non-euclidean geometry, in contrast, draws with *straight* lines—albeit straight lines which might appear to curve to a being who sees and imagines in only three dimensions. (Recall that non-euclidean geometries arises from a denial of postulate five, which is about straight lines, not curved.)

In his masterful work, *Abstraction from Matter*, Charles DeKoninck makes the point that the very necessity of mathematics depends upon the reality of its object. "There would be no geometry as a science if the plane triangle could cease to have its three angles equal to two right angles, or if the diagonal could become commensurate with the side of its square."[31] In a possible allusion to spherical geometry, or non-euclidean geometries—or both—DeKoninck distinguishes between the Euclidean account of geometry conducted on a sphere and the wholly different subject-matter of non-euclidean geometries:

> It may be objected that when the triangle is bent over a sphere the angles become together greater than two right angles, so that this property of the triangle is variable. They who choose to see things in this way should logically hold that, since what is true of the spherical triangle is not true of the plane triangle, what was seen to be true of the latter has now come to be false of it. Furthermore, the spherical triangle is not arrived at by bending the flat one over a sphere, as if it were flexible, no more than a straight line becomes a curve when we draw a line that is the shortest distance between two points on the surface of a sphere, for this is a new kind of shortest distance between two points, *just as the spherical triangle is a new kind of triangle that leaves the flat one unchanged*. There would have been an error had we assumed that what is true of the plane triangle should also be true of the spherical one. As to wilting triangles, they can be handled by the application of calculus.[32]

These two instances serve to illustrate some of the difficulties one encounters when asking whether the ancient framework for *episteme* first discovered by Aristotle and elaborated upon by Thomas Aquinas—and many others—can accommodate historical developments in mathematics.

[31] DeKoninck, "Abstraction from Matter," 170.
[32] Ibid., footnote 2. Emphasis mine.

We have seen that, for modern developments in number theory, either we must extend the term *number* itself beyond its original meaning or we need to exclude these subsequent constructions from being legitimate concerns of mathematics, as *episteme*. At issue is whether, on the one hand, such developments do obtain in the real world which, once again, is foremost among the conditions Aristotle lays down for *episteme*, and on the other, whether the particular *way* in which mathematics considers its objects—its method—can be consistently extended to such things.

We wish to close this chapter with an observation we made earlier, near the end of Chap. 6, when considering numerical extensions demonstrably inconsistent with the first principles of arithmetic. If *episteme* is to be found among these extensions to traditional geometry—we mean topological relations and non-euclidean geometries—it will be in conformity to some real principle. That principle *may not be* continuous quantity in three dimensions. Moreover, the application and utility of topology and non-euclidean geometries—even if they do rise to the level of *episteme*—indicates that there must be *some* sort of truth to be found there. As we said above, not all speculative truths are sciences and not all truths are speculative. In a manner analogous to Kronecker's famous declaration, if we were to we conclude that God made the world a three-dimensional *continuum* and all else is a human construct, even so, there is some reason why we would say such constructs are not mere fictions.

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CHAPTER 11

Cantor, Finitism, and Twentieth-Century Controversies

Finitism and Infinitism

We now return to an account of Georg Cantor's *transfinite* numbers, of which we made much earlier, focusing at that time upon the philosophical questions it raised. Recall that Cantor's attempts to justify transfinites were directed to mathematicians and philosophers alike—both ancient and contemporary. He was well-aware of objections to actually infinite quantities coming from the philosophical quarter—in particular with those of Aristotle—as well as of the mathematical and philosophical climates of his own day.[1] His first issue was whether a workable mathematics of real and actually infinite quantities survives the traditional critique. On the other hand, it was Cantor's contention that the transfinites are not only consistent and applicable, but real.[2] He was unusual among his colleagues in this

[1] Cantor sees Aristotle as the source of the scholastic rejection of actual infinities. He places his counter-arguments to Aristotle's objections in Part 5 of *On Infinite, Linear Point Manifolds*. Cf. Robin Small, "Cantor and the Scholastics," *American Catholic Philosophical Quarterly* 66 (1992): 407–28 and Dauben, *Georg Cantor*, 120–2.

[2] By *real* Cantor means (1) as including the rationals and irrationals, (2) as consistent and definite ideas existing in the mind, and (3) as existing outside the mind. Moreover, all numbers which are real in sense (2) are images of realities in sense (3). See Cantor, "On Infinite, Linear Point-Manifolds," § 8 and Dauben, *Georg Cantor*, 125–6.

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extra-mathematical interest.[3] As it turned out, the *philosophical* issues surrounding infinite quantities figured prominently in subsequent mathematical developments.[4] We plan to use our review of Cantor as an exploration into the genesis of the three most prominent philosophies of mathematics of recent times: formalism, logicism, and intuitionism. We hope to supply an account, not only of how Cantor's claims led to this great divide in mathematical thought, but also how some of the prominent difficulties faced by the modern schools may be avoided *via* the Aristotelian and Thomistic account we have presented.

Cantor differentiates the *potential* infinite—an unendingly increasing (but at any point finite) variable[5]—from the *actual* infinite, which he sees as analogous to an infinitely distant yet definite point. Unlike the variable, and like the point, *transfinite* numbers are fully *determinate* yet *infinite*, having well-defined relationships to one another and being subject to their own mathematical laws.[6]

Up to Cantor's day, one might have tended to think that, even if there *were* such a thing as infinity, there could only be one, since infinity always equals infinity. Cantor's discovery amounts to this: that one can *distinguish* determinate infinities from one another. Mathematicians need only construct them to observe their distinctions.[7]

Cantor uses three *principles of generation* to create the transfinite numbers.[8] The first principle adds unity to a given integer, thereby creating any finite integer. As he considers the unending series of finite integers (I), however, he realizes that the number of members in this class is infinite and there is therefore no greatest number in the series. These observations lead him to the second principle of generation.

[3] While many of his mathematical contemporaries ignored these philosophical concerns and came to defend transfinites, they eventually became the main focus of Cantor's thoughts, with mathematics proper becoming secondary. See especially Dauben, *Georg Cantor*, 297–9.

[4] Cantor, "On Infinite, Linear Point-Manifolds," §§ 1, 8. See also Dauben, *Georg Cantor*, 125–6.

[5] Cantor, "On Infinite, Linear Point-Manifolds," § 1.

[6] Ibid. See also Dauben, *Georg Cantor*, chapter 6.

[7] Constructive discovery, not creation. Cantor holds that the transfinites already exist in the divine mind. Cf. Small, "Cantor and the Scholastics," 408, note 7 and Dauben, *Georg Cantor*, 228–9.

[8] Ibid. The actual application of these principles of generation and the creation of the number classes occurs in § 11. Similar accounts of the generation of such numbers can be found in Russell, *Introduction to Mathematical Philosophy*, chapter 9 and Hilbert, "On the Infinite," 191.

> However contradictory it might be to speak of a greatest number of class (I), there is nevertheless nothing offensive in thinking of a new number which we shall call ω, and which will be the expression for the idea that the entire assemblage (I) is given in its natural, orderly succession. (Just as ν is an expression for the idea that a certain finite number of unities is united to form a whole.) It is accordingly permissible to think of the newly created number ω as the limit to which numbers ν approach, if by it nothing else be understood than that ω is the first integer which succeeds all numbers ν, that is, it is to be regarded as greater than every one of the numbers ν.[9]

If one conceives of the series of finite integers—not of a single integer but of the set as a whole—the limiting number of this series must be greater than any one of them. But since any integer in such a set is finite, that which is greater than any such number must be infinite. In this way is derived the first infinite ordinal number, ω. It is both *the first integer which succeeds all numbers ν* and *greater than every one of the numbers ν*.[10] The process of producing ω is radically different from the process of adding a unity, first mentioned.

> If any definite succession of definite real integers is given for which no greatest exists, a new number is created on the basis of this second principle of generation, which is to be thought of as the limit of those numbers, that is, to be defined as the next greater number to all of them.[11]

Given a definite and unending sequence of integers, the generation of a limiting integer is warranted—an integer which is itself not finite but infinite (and so clearly not a member of that series).

At this point, however, Cantor has shown that the limiting number for the set of all finite integers was an infinite number which he called ω, something which might be confused with the potentially infinite, symbolized by ∞. To show that ω really differs from ∞, Cantor has to prove that it is a *definite* number, standing in relation to other definite numbers of

[9] Cantor, "On Infinite, Linear Point-Manifolds," § 11, 131–132. Cantor's own note here states: "The symbol ∞ which I used in Number 2 of this series of articles (Vol. XVII, p. 357) is hereby replaced from now on by ω, since the sign ∞ has been frequently used as a symbol of indefinite infinities."

[10] Ibid, p. 132. Note that ω is defined as a number succeeding the set of all finite integers which is not itself a member of that set.

[11] Ibid.

the same sort and governed by definite arithmetical laws.[12] In short, Cantor has first to show that there are other numbers like ω—other infinite numbers—which are clearly distinguishable from it.

Having produced ω, Cantor next applies the *first* principle of generation to it, thereby creating ω + 1, ω + 2, and so on, that is, a series of (now) *infinite* integers of which there is no greatest. Since the conditions for applying the second generational principle are met yet again,[13] a new number is created, 2ω, to which is applied the first principle in turn, yielding 2ω + 1, 2ω + 2, and so on.[14] Not only does Cantor generate infinite numbers, he also generates them in classes, subsequently named the *first number* class, the *second number* class, and so on—the first number class being the unending series of finite integers (I), the second being the unending series ω, ω + 1, ω + 2, and so on, the third being 2ω, 2ω + 1, 2ω + 2, and so on. Application of the first principle generates numbers within a number class and application of the second advances to the next number class. The possibilities, Cantor triumphantly exclaims, are without limits.[15] As to their properties, the new infinite numbers are definite—essentially *unlike* ∞—properly arranged, and infinite in number.

What remains is to show that his discoveries are governed by laws—albeit differing, at least in part, from laws governing the finite integers, including the fundamental and intuitively certain laws of association and commutation.[16] While the associative law for addition [a + (b + c) = (a + b) + c] holds for all the integers—finite and infinite—the commutative law for addition (a + b = b + a) holds for the finite numbers alone. 1 + ω does not equal ω + 1, for 1 + ω = ω, while ω + 1 is the second of the infinite ordinals—as we have seen. Similarly, the commutative law for multiplication (ab = ba) holds for finite, but not infinite, numbers and the associative law of multiplication [a(bc) = (ab)c] holds for both.[17]

[12] These conditions for distinguishing the potential from the actual infinite were laid out by Cantor in § 1 of "On Infinite, Linear Point-Manifolds".

[13] Namely, there was a series of definite real integers of which there was no greatest.

[14] There follow in Cantor"s account the generation of two new number classes, of which ω^2 and ω^ω are the first members.

[15] Georg Cantor, "On Infinite, Linear Point-Manifolds," 133.

[16] As it turns out, this sameness and difference serves Cantor's purposes well. That his new integers are governed by (some of) the same laws as other integers strengthens his claim that they are integers, whereas the need to modify existing mathematical laws strengthens his claim that they are new.

[17] Cantor, "On Infinite, Linear Point-Manifolds," §3. Other surprising differences between finite and infinite arithmetic are documented in Russell, *Introduction to Mathematical Philosophy*, chapters 8–9, especially 86–7 and 94–5.

With these series of real integers, finite or not, Cantor associates a number of a different sort, and a theory of finite and infinite *powers* arises.[18] The notion of infinite powers, unlike the ω-numbers, corresponds to the size, or quantity of such sets, independent of the ordering of the elements. Two sets have the same power, so defined, if their elements can be placed in a one-to-one correspondence with each other, independently of their ordering.

The power of a finite set, says Cantor, is the same as its ordinal number, but the power of infinite sets—and in this Cantor identified what he believed to be the essential difference between finite and infinite sets—remains the same, despite ordinal changes.[19] The ordinal number of (I), for example, varies depending upon the principle of order used. To the set of finite real integers: 1, 2, 3, … ν, … corresponds ω, the first infinite ordinal. However, to the set of finite real integers: 1, 3, 4, … ν + 1, 2… corresponds ω + 1, the second infinite ordinal. Similarly, to the well-ordered set of finite real integers: 1, 3, 5, … ν, … 2, 4 corresponds ω + 2, and so on. The *same* infinite series can be measured by different infinite ordinal numbers, subject to different orderings. Yet any ordering of the elements of this series produces the same *power*. Of these infinite powers, $\aleph_0$, the power of the set of all finite integers, is demonstrably the least.

Cantor next shows that a set having the power of one class (say, the class of finite integers) would have ordinal numbers belonging to the next higher class (in this case, ω, ω + 1, ω + 2, and so on). Not only were there many definite and distinct infinite numbers and distinct classes of them, as well as arithmetical laws which partially differed from those of finite numbers, there were also different types of infinite numbers altogether (ordinals and cardinals), having definite relationships each to the other.[20]

[18] Cantor, "On Infinite, Linear Point-Manifolds," §1. Cantor seems to mean by *power* what we mean by *cardinal number*.

[19] Ibid., §2. In fact, said Cantor, the set can be assigned any ordinal numbers in its corresponding number series, simply by changing this principles of ordering: "Every set whose power is of the first class is countable by numbers of the second number class and only by such, and it is always possible to give the set a succession of elements that is countable by any arbitrarily chosen number of the second number class." Ibid., §2, 98. Cantor notes that the set must be a *well-ordered* one: there must be a first element in the set, and every other element (excepting, perhaps, the last) must followed by a determinate element, which is next.

[20] Cantor, "On Infinite, Linear Point-Manifolds," § 2. For more detailed summaries of these relationships and their significance, see Russell, *Introduction to Mathematical Philosophy*, especially chapters 8–9, Hilbert, "On the Infinite," and Dauben, *Georg Cantor*, especially chapters 4–5.

Cantor had provided the mathematical community with more infinity than they could possibly have expected.[21]

Mathematicians had varied reactions to Cantor's published claims, from Bertrand Russell and David Hilbert, who defended them, to Ernst Kronecker[22] and Henri Poincaré, who strongly opposed them.[23] As we noted above, Cantor was well aware of the inevitable objections his theory of transfinite numbers would raise among mathematicians and philosophers alike. Despite the sympathetic hearing of some, Cantor thought the philosophical objections simply had to be addressed. Though granting that the forthcoming justification of transfinite numbers was not a specifically mathematical obligation, he nevertheless felt compelled to deal with the objections all the same.[24] The properly philosophical questions, Cantor saw, were of great importance even for mathematics itself. In truth, as the implications of Cantor's work became clearer, Frege noted that the issue would result in a damaging internal conflict.

> Here is the reef on which it [mathematics] will founder. For the infinite will eventually refuse to be excluded from arithmetic, and yet it is irreconcilable with that [finitist] epistemological direction. Here, it seems, is the battlefield where a great decision will be made.[25]

The objections began to surface, then, and other mathematicians joined the fray. Frege's analogy of a battle was becoming all-too-true. Most famous, perhaps, is David Hilbert's war-cry: "No one shall drive us out of the paradise which Cantor has created for us."[26] It is to his credit that Cantor anticipated the revolutionary character of his discoveries. What remained was whether he would successfully address the very root of many of the objections to his theory, Aristotle himself.

[21] Hilbert once described transfinite numbers as the "paradise which Cantor has created for us."

[22] Kronecker was Cantor's contemporary and a noted *finitist* mathematician. Apparently, Kronecker's criticism of transfinite theory was instrumental in leading Cantor to write the more philosophical sections of "On Infinite, Linear Point-Manifolds." Their ongoing debate concerning infinity was often quite acrimonious. See Dauben, *Georg Cantor*, 13–138, for example.

[23] Dauben, *Georg Cantor*, 1.

[24] Cantor, "On Infinite, Linear Point-Manifolds," §8.

[25] Quoted in Dauben, *Georg Cantor*, 225.

[26] Hilbert, "On the Infinite," 191.

There is another, albeit secondary, legitimization of developments in number theory to which Cantor does give some weight, that is, its successful application to the physical world.

> There is no need, as I believe, of fearing any danger to the sciences in these principles, as many think: the conditions imposed under which the freedom of constructing numbers can be exercised are of such a kind that they permit very little room for arbitrary action (*Spielraum*); for every mathematical concept carries within itself the necessary corrective; if it is fruitless and purposeless, that is soon seen and it will because of little success be abandoned.[27]

Cantor instances the "theory of functions" of Gauss, Cauchy, and others, "…which, although created and erected *in full freedom without transient purpose* nevertheless reveals its transient meaning in applications to mechanics, astronomy, and mathematical physics, as was to be expected."[28] So, too, the theory of differential equations he ascribes to Fuchs and Poincaré and the "ideal" numbers introduced by Kummer into number theory. By Cantor's account, then, the place of metaphysics in mathematics is not to set *a priori* limits to the development of mathematically consistent systems of thought, but to establish the *transient* reality of concepts already shown to be real.

As to his own specific contribution to number theory, Cantor sees a need not only to discover the relationships obtaining among his newly discovered infinities, but to address traditional and modern objections to the very *notion* of infinite quantities, the largest part of which come from philosophers.[29] A considerable portion of the *Grundlagen* is devoted to addressing these objections.[30]

[27] Cantor, "On Infinite, Linear Point-Manifolds," 115.

[28] Ibid., 116. Emphasis mine.

[29] Note that such objections do not consist in denying transient correlates in the physical and intellectual world (unless secondarily), but bring the charge of inconsistency directly to the system of transfinite numbers. Among his opponents, Cantor saw Aristotle as being the first. See Dauben, *Georg Cantor*, chapter 5.

[30] The mathematical portions of Cantor's "On Infinite, Linear Point-Manifolds" deal with the establishment of infinitely large numbers. In section four, Cantor raises, considers, and denies the possibility of a similar extension into the infinitely small. In the course of this discussion, he considers yet another view which "…denies the actual existence of any other numbers than the integers of our number-class (I)." Cantor, "On Infinite, Linear Point-Manifolds," 103. It is here, in sections four and five, that Cantor raises and addresses so called *finitist* objections to his transfinite theory, beginning with Aristotle.

Reactions ranging from Hilbert's to Kronecker's "God made the integers; all else is the work of man"[31] lead us to ask: why such a divergence?

Cantor himself acknowledges that some think mathematical freedom a dangerous notion. *Finitists* see developments declared legitimate on the basis of consistency alone as being subject to *error*.[32] Some are concerned that Cantor's consistent systems may only be *real* in the established sense of the word—definite and distinguishable from other mathematical notions with which they have been shown to be in accord. For his part, Cantor insists that these objects have *transient* correlates (and so are *real* in a second sense also), but even he admits that discovering such connections, the slow, hard work of metaphysics, is not a prior condition of mathematical legitimacy. Presumably, such correlations might *never* be established.

That there *are* prominent objections to infinite numbers we have already directly seen in Aristotle's own words. Section four of Cantor's Grundlagen contains his consideration of Aristotle.[33]

Cantor first notes that some of those who deny transfinite numbers would admit the existence of the rationals, since they come directly from the integers and are expressed in terms of the integers.[34] He goes on to say, though, that traditional mathematicians are squeamish when it comes to the irrational numbers.

> The actual material of analysis is composed, in this opinion, exclusively of finite, real integers and all truths in arithmetic and analysis already discovered or still to be discovered must be looked upon as relationships of the finite integers to each other; the infinitesimal analysis and with it the theory of functions are considered to be legitimate only in so far as their theorems are demonstrable through laws holding for the finite integers.[35]

[31] Quoted in Weber, "Leopold Kroenecker," 19.

[32] Cantor, "On Infinite, Linear Point-Manifolds," §4, 553–4. It is on the reef of mathematical truth that this particular controversy founders.

[33] Once again, Cantor sees Aristotle's arguments against the actual infinite as the origin of the scholastic principle *infinitum actu non datur*.

[34] As we saw, the negative integers can be considered an extension of the integers, for example, by allowing subtraction in every case.

[35] Georg Cantor, "On Infinite, Linear Point-Manifolds," § 4, 103. (I have substituted *legitimate* for *legalized* in Bingley's translation.) Note that a relationship to finite integers is here taken as the source of mathematical legitimacy.

While admitting the benefits such a view brings to mathematics, Cantor finally discounts it as erroneous and overly restrictive.[36] Not all numbers, then, can be reduced to the finite integers.

Having raised the issue, Cantor takes up Aristotle himself, citing *Metaphysics* XI as his reference for Aristotle's arguments against the infinite.[37] Cantor specifically addresses two of Aristotle's arguments and then turns to the *source* of Aristotle's error.[38] The general problem, he claims, is that Aristotle begs the question—he assumes that all numbers must be countable by means of finite numbers and thereby proves that infinite numbers are not numbers.

> If one considers the arguments which Aristotle presented against the real existence of the infinite (vid. his Metaphysics, Book XI, Chap. 10), it will be found that they refer back to an assumption, which involves a *petitio principii*, the assumption, namely, that there are only finite numbers, from which he concluded that to him only enumerations of finite sets were recognizable.[39]

Cantor is referring to Aristotle's argument that an actually infinite number is impossible, since every number, or whatever has a number, can be numbered.[40] Of course, the actually infinite cannot be numbered in this way, that is, one cannot enumerate the members of an infinite number one at a time.[41] This, says Cantor, is at the heart of finitists' arguments against

[36] Such a requirement does enable one to avoid the *dangers* associated with certain types of mathematical speculation, where *anything is possible*. Still, it does so at the expense of mathematical freedom. See Cantor, "On Infinite, Linear Point-Manifolds," §4.

[37] Aristotle's arguments against the infinite in book eleven of *Metaphysica* are excerpted from the last five chapters of book three of *Physica*, his *ex professo* treatment of the infinite.

[38] The arguments themselves are taken up in §4 of "On Infinite, Linear Point-Manifolds," while the source of the difficulty is dealt with in §6.

[39] Georg Cantor, "On Infinite, Linear Point-Manifolds," §4, 104–105.

[40] *Metaphysica*, XI, 10, 1066b26–27. "ἀριθμητὸν γὰρ ὁ ἀριθμὸς ἢ τὸ ἔχον ἀριθμόν."

[41] The argument Cantor presents is a *verbatim* excerpt from *Physica*, III 5 204b8, where it is followed by: "Now if what can be numbered is able to be numbered, one will also be able to go through the infinite." εἰ οὖν τὸ ἀριθμητὸν ἐνδέχεται ἀριθμῆσαι, καὶ διεξελθεῖν ἂν εἴη δυνατὸν τὸ ἄπειρον) Aquinas expands the argument in *In Physicorum*, III 8 351: "Everything numerable can be numbered, and consequently is gone through by numbering; but every number, and everything which has a number, is numerable, and so everything of this kind can be gone through. If, then, some number were infinite (whether it be separate or existing in matter,) it would follow that it is possible to go through the infinite; but that is impossible." "Omne enim numerabile contingit numerari, et per consequens numerando transiri; omnis autem numerus, et omne quod habet numerum, est numerabile; ergo omne huiusmodi contingit transiri. Si igitur aliquis numerus, sive separatus, sive in sensibilibus existens, sit infinitus, sequetur quod possibile sit transire infinitum; quod est impossibile."

the infinite: for they expect infinite numbers to have the same properties as do finite numbers.[42] On the other hand, and to their credit, neither Aristotle nor Thomas Aquinas take the argument in question as having much weight.[43] The argument is merely probable, says Aquinas, since it does not have a necessary conclusion. Moreover, he *anticipates* what someone defending infinite numbers (like Cantor himself) might say in response.

> These arguments are probable and proceed from things which are commonly said. For they do not conclude of necessity: for... someone who said that some multitude is infinite would not say that it is a number or that it has a number. For *number* adds to *multitude* the notion of a measure: for number is a multitude measured by the unit, as is said in the tenth book of the *Metaphysics.* And because of this number is said to be a species of discrete quantity, but not multitude, which pertains to the transcendentals.[44]

Cantor insists upon keeping certain elements of the traditional notion of number when speaking of his transfinite numbers, yet he clearly wishes to dissociate from them the fundamental notion of being *able to be gone through*, of being *enumerable* in the traditional sense. Cantor sees Aristotle as allowing his first principle of generation alone when it comes to mathematics. It would be no surprise, then, if such a one would see the infinite

[42] Cantor, "On Infinite, Linear Point-Manifolds," §6.

[43] When presented in *Physica*, the argument is described by Aristotle as proceeding logically, a manner which is contrasted to proceeding physically, or according to the principles of his physics. *Physica*, III 5 204b4, 10. "λογικῶς μὲν οὖν σκοπουμένοις ἐκ τῶν τοιῶνδε δόξειεν ἂν οὐκ εἶναι … φυσικῶς δὲ μᾶλλον θεωροῦσιν ἐκ τῶνδε." Aquinas explains the difference: "He calls these first arguments logical, not because they proceed logically from the terms of logic, but because they proceed in the manner of logic, that is, from things which are common and probable, which is proper to the dialectical syllogism." *In Physicorum*, III 5 349. "Dicuntur autem primae rationes logicae, non quia ex terminis logicis logice procedant, sed quia modo logico procedunt, scilicet ex communibus et probabilibus, quod est proprium syllogismi dialectici." The Ross edition of Aristotle uses the word *dialectical* to describe the argument.

[44] *In Physicorum*, III 8 352. "…istae rationes sunt probabiles, et procedentes ex iis quae communiter dicuntur. Non enim ex necessitate concludunt: quia … diceret aliquam multitudinem esse infinitam, non diceret eam esse numerum, vel numerum habere. Addit enim numerus super multitudinem rationem mensurationis: est enim numerus multitudo mensurata per unum, ut dicitur in X Metaphys. Et propter hoc numerus ponitur species quantitatis discretae, non autem multitudo; sed est de transcendentibus."

in quantity as essentially indeterminate and ultimately infinite in a merely potential way, as something similar to Cantor's *ideal* infinity, mentioned above. If the transfinite numbers are in fact real, Aristotle's argument would certainly be subject to the charge of question-begging. For one might ask how Aristotle knows that there are no infinite numbers, to which his response, as Cantor reports it, would simply be *because all numbers are finite*, which, of course, is the point in question.

Still, and even as Cantor presents it, there is something more to Aristotle's argument, namely, the claim that numbers are *numerable*. It is not so much Aristotle's claim that all numbers are finite which Cantor attacks here, but the notion that all enumerations are finite ones, or are made in terms of finite numbers, or—most precisely—that one can make determinate (or measure) the quantity of some number using finite numbers alone. The charge of *petitio principii* stands only on the supposition that one could show that not all numbers are finite ones. Cantor is well aware of this.

> I believe that I have proven above, and it will appear even more clearly in what follows in this paper, that *determinate enumerations of infinite sets* can be made just as well as for finite ones, assuming that a definite law is given the sets by means of which they become well-ordered.[45]

Despite Aristotle, infinite sets can be measured, says Cantor, and their elements can be enumerated. They are not measured, however, in the way finite sets are, by means of the unit and in terms of some finite number of such units. Rather, they are measured by means of a limiting number which succeeds all in the set and which is greater than any finite quantity of units.[46] Such a number is generated—*discovered*, Cantor would say—by means of a principle which is altogether different from that which generates enumerations among finite numbers. This infinite enumeration depends upon the principle whereby the members of the set being numbered are ordered; enumerations of finite sets, though they also require an ordering, are the same despite the particular ordering chosen.[47] Members of the higher number classes differ from themselves in determinate ways

[45] Cantor, "On Infinite, Linear Point-Manifolds," §4, p. 105. Emphasis mine.

[46] Recall Cantor's second principle of generation, above.

[47] The essential difference between the finite and the infinite, says Cantor, is the simple fact that infinite sets have infinitely many ordinal numbers whereas finite sets have only one. Cantor, "On Infinite, Linear Point-Manifolds," §4.

acceptable to mathematicians (ω + 1 is one greater than ω) and laws of arithmetic can be formulated to deal with these numbers. Thus, Cantor would say, not all numbers are measured by counting up units, but only finite ones. Infinite numbers are measured by correlating a series of arithmetically different infinite integers with certain orderings of their members, such that a different ordering yields a different enumeration. These enumerations, being determinate, are a legitimate subject for a mathematician's consideration.

> The assumption that besides the Absolute (which is not obtainable by any determination) and the finite there are no modifications which, although not finite, nevertheless are determinable by numbers and are therefore what I call the actual infinite—this assumption I find to be thoroughly untenable as it stands.[48]

The second Aristotelian argument is that "the finite would be dissolved and destroyed by the infinite if it (the infinite) existed, since the finite number is allegedly destroyed by the infinite."[49] As presented, the argument has some cogency. How could one *increase* infinity? How could one go beyond that which is beyond all? In short, how could one infinite be greater than another? The greater implies the less and therefore has a definite limit, or term. If something were less than infinite, how could it *be* infinite? Cantor is well aware that the this argument concludes to: were there an infinite, there could be only one. Nor would it be determinate or, especially, numerable, since to number it (or to measure it) would be to establish some point at which it ended, and beyond which it did not go.

[48] Georg Cantor, "On Infinite, Linear Point-Manifolds," §5, 107.

[49] Ibid., §4, 105. If he has book eleven of the *Metaphysics* in mind, Cantor may be referring to the arguments given in lines 1066a28–32 and 1067a18–21 of chapter 10, though these more obviously deal with an infinitely large universe composed of a finite number of kinds of elements, the result being that one kind of element, being infinite in size, would destroy the others. (Similar arguments appear in book three of the *Physics*, chapter 5, at lines 204b13–19 and lines 205a 24–25.) An analogous case might be made for numbers. Just as the *addition* of a finite amount of water, say, to a universe in which there was already infinite fire would not change the situation, so the addition of a finite number to an infinite number would not result in anything new. As Dauben expresses it: "Given any two finite numbers a and b, both greater than zero, their sum a + b > a, a + b > b. However, if b were infinite, no matter what finite value a might assume, a + ∞ = ∞." Dauben, *Georg Cantor,* 122.

Cantor's response comes down to asking whether an infinite number—though it measures an unlimited number of members in a series[50]—need *itself* be unlimited in Aristotle's sense of the word.[51] As we saw earlier, though any number within the series of finite integers is a limit, in some sense, with respect to those which precede it, ω is not *such* a limit: it exceeds all numbers within the series, but is not itself a member of that series. So, a limit need not be finite. Nor is it the case that an infinite limit be itself without limits, for though it is the limit for integers in the first number class (and so is without a *finite* limit,) ω is itself surpassed by countless integers in its own number class, whose limit is the first number within the third number class (2ω), and so on. There are many numbers, then, which are *beyond* ω, though no finite number can surpass it. It is a limit (and so a definite quantity), and is itself both limited and unlimited, but in different respects.[52]

To return to the argument proper, then, while one could not add a finite quantity to Aristotle's actually infinite number, that is, to a quantity which surpasses all quantities and is strictly unlimited—and in this sense the finite would be destroyed by the infinite—still, Cantor argues, to a *determinate* and *complete* infinite number, a finite number can indeed be adjoined and united without effecting its dissolution. The infinite number is itself modified by an adjunction of a finite number.[53]

What sense does Cantor make of adding a finite number to an infinite number? Recall his qualification of the commutative law for addition when it comes to infinite numbers. According to Cantor, $1 + \omega = \omega$, while $\omega + 1 > \omega$. In fact, $\omega + 1$ is the second of the infinite ordinal numbers. Returning to the example of the series of finite integers mentioned above, let us suppose the well-ordered series of finite integers: 1, 2, 3, … ν, … which has, as its limiting number, the first infinite ordinal ω. Let us now suppose that

[50] Recall that Cantor sees any infinite real integer created by means of the second principle of generation as a limiting *number*. The first infinite ordinal, ω, is seen as the limit of the unending series of finite integers, (I). It is as such a limit that it can be called a *measure* of the number of members in that series.

[51] Aristotle's term for the infinite, τὸ ἄπειρον, can be translated *infinite*, *boundless*, *unlimited*, or even *indefinite*.

[52] Cantor describes his discovery as "that which regards the infinitely great not merely in the form of that which increases without limit … but also which fixes it by numbers in the determinate form of the completed infinite." Georg Cantor, "On Infinite, Linear Point-Manifolds," §4, 106.

[53] Georg Cantor, "On Infinite, Linear Point-Manifolds," §5, 105.

we displace one member of the series, obtaining: 2, 3, 4, … ν + 1, … 1. Since we are dealing with a series in which there is no greatest, and since the numbers in the series are reciprocally well ordered (up to the last element of the second series, that is, to 1 in the first corresponds 2 in the second, to 2 in the first corresponds 3 in the second, to ν in the first corresponds, ν + 1 in the second, and so on), there are as many members up to 1 in the second series as there are in the first series, taking their *ordering* as the operative principle. Therefore, the ordinal number of the second series is one greater than that of the first, or ω + 1.[54] Nevertheless, it is crucial that the finite number be added to the infinite number, and not conversely. For, in comparing the well-ordered series of finite integers: 1, 2, 3, … ν, … to the series 2, 1, 3, … ν, … one can see that there is, in fact, a one-to-one correspondence throughout. In other words, ν + ω = ω (where ν is any finite integer).[55]

> If ω is the first number of the second number class, then 1 + ω = ω, but ω + 1 = (ω + 1), where (ω + 1) is a number entirely distinct from ω. Everything depends, as is here clearly seen, upon the position of the finite relative to the infinite; in the first case, the finite is absorbed into the infinite and vanishes, but if it modestly takes its place after the infinite it remains intact and unites with the infinite to form a new (since modified) infinite.[56]

Finally, Cantor addresses the very heart of Aristotle's difficulties with infinite numbers in general, which Cantor calls the πρώτον ψεύδος—the initial falsehood—upon which all finitistic reasoning is based.

> All so-called proofs against the possibility of actually infinite numbers are faulty, as can be demonstrated in every particular case, and as can be concluded on general grounds as well. It is their πρώτον ψεύδος that from the outset they expect or even impose all the properties of finite numbers upon the numbers in question, while on the other hand the infinite numbers, if

[54] If ordering is not a concern, that is, if we simply establish a one-to-one correspondence between members of the series (i.e., 1 with 1, 2 with 2, and so forth,) the number will be the same. In this case, the series of finite integers has a number of $\aleph_0$, the smallest infinite cardinal.

[55] Russell defines *greater than* with respect to infinite ordinals in this way: "One serial number is said to be 'greater' than another if any series having the first number contains a part having the second number, but no series having the second number contains a part having the first number." *Introduction to Mathematical Philosophy*, 90.

[56] Georg Cantor, "On Infinite, Linear Point-Manifolds," §5, 109.

> they are to be considered in any form at all, must (in their contrast to the finite numbers) constitute an entirely new kind of number, whose nature is entirely dependent upon the nature of things and is an object of research, but not of our arbitrariness or prejudices.[57]

Transfinite numbers are not extensions of the finite integers in the sense that an infinity has been added to a given finite integer to produce them. At the heart of Cantor's charge of question-begging is that a transfinite number is a different *kind* of number. One cannot reasonably require that the properties of one species within a genus be applicable to another within that same genus. Aristotle's basic error was in taking one species of number, the finite integers, to *be the genus itself*, number. One might just as mistakenly take the species of rectilinear figures to be the genus itself, thereby excluding the whole class of curved figures from consideration. But what holds for one among the species within a genus does not necessarily hold for the others:

> It is here [by finitists] tacitly assumed that properties which for numbers as we have previously understood them are disjunct, are equally so for the new numbers, and one accordingly concluded the impossibility of infinite numbers. Who fails to see this fallacy at a glance? Isn't every generalization or extension of concepts associated with the abandonment of certain special properties, even unthinkable without it?[58]

As we have also seen, Cantor likens his introduction of the theory of transfinite numbers to previous extensions of the concept of number, such as the rationals, the irrationals, and the complex numbers.[59] *Such* extensions, he notes, have been regarded as mathematically legitimate.

[57] "Letter of Georg Cantor to Gustav Eneström," quoted in Dauben, *Georg Cantor*, 125.

[58] Georg Cantor, "On Infinite, Linear Point-Manifolds," §6, p. 110. The properties in question are even and odd. The argument would go as follows: since an infinite number is neither even or odd, and since all numbers are either even or odd, then an infinite number cannot be a number. Cantor maintains that the property of being either even or odd belongs to finite numbers, not to all numbers.

[59] The very first sentence of "On Infinite, Linear Point-Manifolds" points out that any advance in the theory of sets depends upon extending the concept of real integer in this way. For more on extensions to the concept of number, see Russell, *Introduction to Mathematical Philosophy*, chapter 7.

> It (mathematics) is obligated when new numbers are introduced to give definitions of them by which such a determinacy and, under certain conditions such a relationship to older numbers is granted them, that they can in any case be definitely distinguished from each other. As soon as a number satisfies all these conditions [consistency, and standing in determinate, orderly relationships to other numbers] it must be regarded as mathematically existent and real. It is in this that I see the reason given in paragraph 4 why the rational, irrational and the complex numbers are to be considered as much existent as the finite positive integers.[60]

Cantor's strategy is to defend transfinite numbers as legitimate extensions of the concept of real integer by establishing their consistency and definite relationships to the finite integers. To that extent, they would then be as legitimate as previous extensions of the same sort and included within a distinct species of number.

The charge of taking the species to be the genus is a serious one, since the mistake would have occurred in the very principles of mathematics, affecting the remainder of that study. Yet how might this have occurred?

Bertrand Russell offers a distinct account of mathematical reasoning in general. He recognizes, of course, the basic distinction between arriving at the general principles of a science and making deductions from such principles. Thus, one might arrive at Euclid's axioms and postulates by generalizing from practices in land-surveying and then deduce other propositions from the principles so discovered. With respect to the question where mathematical reasoning begins, then, Russell gives a surprising answer: "The most obvious and easy things in mathematics are not those that come logically at the beginning; they are things that, from the point of view of logical deduction, come somewhere in the middle."[61] One would not naturally begin mathematical reasoning with the axioms, then, making deductions from these, rather, one might begin with some intermediate proposition, deduce something from it, or, conversely, ask on what principle that proposition itself is grounded.

To apply this to Aristotle's πρώτον ψεύδος, one might imagine him abstracting the notion of finite integer—somewhere in the middle, logically, between the genus, number, and the species of finite integer themselves—and seeing it as *the* starting point of mathematical reasoning, much as one might mistakenly take a theorem to be an axiom. If finite number

[60] Georg Cantor, "On Infinite, Linear Point-Manifolds," §8, 115.
[61] Russell, *Introduction to Mathematical Philosophy*, 2.

is number, then infinite numbers cannot be numbers at all. If Russell is correct, however, to regard what one first arrives at in mathematical reasoning to be logically first also is to neglect another form of mathematical thought altogether. Extending the concept of *number*—originally taken to mean *finite number*—so as to include other types (such as the rationals and the mixed numbers) is simply a historical form of Russell's second type of reasoning.

> We shall find that by analysing our ordinary mathematical notions we acquire fresh insight, new powers, and the means of reaching whole new mathematical subjects by adopting fresh lines of advance after our backward journey.[62]

Perhaps finitists mistakenly insist upon first—that is, ordinary—mathematical concepts as being *logically* first. In defining number as *a multitude measured by the unit*, Aristotle is associating properties belonging to a single species of number with the genus, taking finite numbers to be the genus of number itself. The fundamental error in Aristotle's arguments against the actual infinite, then, would be a logical one.[63]

Finally, though, Cantor and Aristotle are speaking of different things. The actual infinite Cantor defends is not the actual infinite Aristotle denies. For one thing, Aristotle takes the *actual* infinite as being wholly unbounded. It is *entirely* without limits.[64] This is why arguments such as that given against a finite number of elements in an infinitely large universe have the cogency they do. Cantor's transfinite numbers, on the other hand, are not wholly unlimited: for since infinite numbers are generated in

[62] Ibid.

[63] One would do well to contrast Russell's account of mathematical learning with Aristotle's own account of learning in the opening passages of the *Physics*. In reasoning, to proceed from the more known to the less known requires that one begin with the more universal and proceed to the less universal. It would be impossible, then, to begin with a species, strictly speaking. An understood species must be seen as a species of some genus. If Aristotle rightly understood finite number, then, he would have to have a right understanding of number also. Note that Cantor does not claim that Aristotle misunderstood the nature of finite numbers, only that he took such to be the genus. What seems to be the case, rather, is that infinite numbers and finite numbers are not species within a common genus.

[64] As he says in *Physica*: "For a body is something having extension on every side, while the infinite is an extension without limits, so that an infinite body would be extended infinitely in every direction." III 5 204b20–22 "σῶμα μὲν γάρ ἐστιν τὸ πάντῃ ἔχον διάστασιν, ἄπειρον δὲ τὸ ἀπεράντως διεστηκός, ὥστε τὸ ἄπειρον σῶμα πανταχῇ ἔσται διεστηκὸς εἰς ἄπειρον."

classes, such that one advances from the second number class[65] to the third by applying the second principle of generation, numbers within such classes are arranged as a series. So, the numbers of the first number class are: ω, ω + 1, ω + 2, and so on. Clearly, any of these is limited: ω + 1 > ω, and so ω + 1 is a sort of limiting quantity for ω. Further, the first number of the third number class, 2ω, is set down as the limiting number for all numbers in the second number class. Therefore, there is always a number greater than any transfinite number, or no transfinite number has an *unlimited* quantity. This accords with what Cantor himself says.

> What I declare and believe to have demonstrated in this work as well as in earlier papers is that following the finite there is a transfinite (*transfinitum*)—which might also be called supra-finite (*suprafinitum*), that is, there is an unlimited ascending ladder of modes, which in its nature is not finite but infinite, but which can be determined as can the finite by determinate, well-defined and distinguishable numbers.[66]

Aristotle's arguments against an actually infinite number were not directed against transfinite numbers as Cantor describes them. Only a number which exceeded all limits would *absorb* finite numbers added to it; transfinite numbers do not, precisely because they are limited in that respect.

On the other hand, perhaps Aristotle's arguments apply to *any but finite* numbers. For Aristotle sees number as a type of quantity, predicated in answer to the question *how much* or *how many*. How to answer such questions? Among discrete quantities, by *counting*. To determine attributes of things in other ways is to ask (and answer) other sorts of question about them, such as *where*, *when*, *of what sort*, and so on.[67] Extensional (enumerative) descriptions differ from intensional ones, that is, *via* specifying some proper characteristic. But in the case of infinite numbers enumeration is not possible and we are left with intensional definitions alone.[68] The difficulty is that intensional definitions seem to belong more properly

[65] Recall that the second number class is the first class containing infinite numbers. The first number class is the class of finite integers.

[66] Georg Cantor, "On Infinite, Linear Point-Manifolds", §5, 107.

[67] The names of several of Aristotle's categories as given in the Greek are interrogatives such as these. For example, ποσὸν, how much or how many, quantity, ποῖον, of what sort, quality, ποῦ, where, place, and ποτὲ, when, time.

[68] Russell, *Introduction to Mathematical Philosophy*, 12–13.

to questions and manners of answering questions which *differ* from the *how much* or the *how many*. In truth, it is clear that an intensional definition may be given independently of any considerations concerning the *how much* or the *how many*. The first infinite ordinal number, ω, may say something about the unending series of finite integers, but it is not clearly an answer to the question of *how many* such integers there are.[69] Perhaps it is more akin to *of what sort* or *in relation to what*. If defining by extension, that is, counting, is how one answers the question *how many*, then properly speaking only finite numbers could be found in the category of quantity—the category containing answers to that question. To insist that transfinites are a species alongside finites is to see them as *numbers* in the same sense. Going outside the very category of quantity to answer other kinds of questions is to equivocate on the term.

Even apart from this, there is the issue of whether Cantor has in fact established that there are such things as transfinite numbers. For Cantor, existence is of two sorts.[70] The sort with which mathematicians *per se* are concerned he calls *intrasubjective* or *immanent* existence. For mathematicians are concerned, not with *transsubjective* reality—what is actually found "in corporeal and intellectual nature"[71]—but with consistency and with determinate relations among mathematical concepts in the mind.

> Mathematics, in the construction of its ideas, has *only* and *solely* to take account of the *immanent* reality of its concepts and has *no* obligation whatever to make tests for their *transient* reality.[72]

It is on account of this distinction that mathematics deserves the name *free mathematics*.[73] Even so, Cantor consistently insists that transfinite numbers are real in the second sense also, that the concepts in the mind are tokens of separate natural or intellectual realities. In truth, Aristotle's arguments against an actual infinite would pose no threat to Cantor's theory unless he were making such a claim also: for the arguments are clearly directed, not against the logical consistency of such concepts, but against their existence in the world.

[69] The same would hold for infinite cardinals.
[70] See §8 of "On Infinite, Linear Point-Manifolds."
[71] Georg Cantor, "On Infinite, Linear Point-Manifolds," §8, 114.
[72] Ibid., 115.
[73] Ibid.

Transfinites are well-defined[74] and so have existence in the mind. As to their *transsubjective* reality:

> Reality can be ascribed to numbers in so far as they must be taken as an expression or image of the events and relationships of that outer world which is exterior to the intellect, as, for instance, the various number-classes (I) (II) (III) etc. are representative of powers which are actually found in corporeal and intellectual nature.[75]

Also, in the same place:

> In lieu of the thoroughly realistic but at the same time none the less idealistic basis of my considerations, there is no doubt in my mind that these two spheres of reality [*intrasubjective* and *transsubjective*] are always found together, in the sense that a concept said to exist in the first way always also possesses in certain and even in an infinity of ways a transient reality.[76]

His reason for *this* claim—which he immediately provides—takes Cantor the mathematician-turned-philosopher of mathematics to another level altogether. For it is the inseparable unity of all things which grounds his claim. "The connection of both realities has its peculiar foundation in the unity of the All, *to which we ourselves belong*." He does not claim that the extra-mental existence of numbers is a thing easy to grasp[77]—more the contrary—still, he is confident that they are there.

Cantor made clear advances in establishing the legitimacy of transfinite numbers as concepts. They appear to be clearly distinct from the finite numbers and from the potentially infinite. They have a definite character and stand in definite relationships both to each other and to other numbers, including finite numbers. They are more-or-less subject to mathematical laws. Even so, the transfinites have not been entirely free of

[74] Cantor's three conditions for the *intrasubjective* reality of our ideas are that they: (1) occupy an entirely definite place in our understanding on the basis of definitions, (2) can be precisely differentiated from all other parts of our thought, and (3) stand in determinate relationships to those parts, and so have a determinate effect on our thought. Ibid., 114.

[75] Ibid.

[76] Ibid.

[77] Determining what the transient reality of such things is "becomes for the most part one of the most troublesome and profound problems of metaphysics and must frequently be left to times in which the natural development of one of the other sciences eventually reveals the transient meaning of the concept in question." Ibid.

difficulty.[78] Moreover, even granting the free use of transfinites at the conceptual level, does it follow that what one conceives of in this way thereby exists?

Among the arguments given in favor of the infinite, Aristotle notes one for which he claims a special status.

> Most important of all [among the reasons for a belief in the infinite] is one which raises a difficulty for everyone: for it seems that number and mathematical magnitudes and what is outside the heavens are infinite because they do not cease in our thought.[79]

Aristotle addresses this argument near the end of his account of the infinite, noting that thinking and what thinking is about may not correspond.

> To trust to thinking is absurd, for the excess or the deficiency is not in the thing but in the thought. For one of us might think that someone is bigger than he is, increasing him *ad infinitum*: but it is not because something thinks this that he is bigger than we are, rather, it is because he *is* [bigger], and the thought is accidental.[80]

Number is real because there are a multiplicity of things which are numbered, which is why Aristotle calls the infinity of number a *potential* one, one consequent upon the division of continuous quantity. By one act of division I produce two things, by two acts three, and so on. Since the *continuum* is divisible *ad infinitum*—but never all at once—the number which is consequent upon such a kind of division is infinite in the same

[78] One would naturally take into account here the discovery of various paradoxes belonging to the very set theory Cantor sought to advance by the introduction of infinite numbers, as well as David Hilbert's attempt to rescue mathematics through formalism, and the consequent, and disappointing, discovery of Kurt Gödel. We will be turning shortly to twentieth-century finitist/infinitist controversies and the schools of thought they produced.

[79] *Physica*, III 4 203b22–25. μάλιστα δὲ καὶ κυριώτατον, ὃ τὴν κοινὴν ποιεῖ ἀπορίαν πᾶσι· διὰ γὰρ τὸ ἐν τῇ νοήσει μὴ ὑπολείπειν καὶ ὁ ἀριθμὸς δοκεῖ ἄπειρος εἶναι καὶ τὰ μαθηματικὰ μεγέθη καὶ τὸ ἔξω τοῦ οὐρανοῦ.

[80] Ibid., III 8 208a14–19. τὸ δὲ τῇ νοήσει πιστεύειν ἄτοπον· οὐ γὰρ ἐπὶ τοῦ πράγματος ἡ ὑπεροχὴ καὶ ἡ ἔλλειψις, ἀλλ' ἐπὶ τῆς νοήσεως. ἕκαστον γὰρ ἡμῶν νοήσειεν ἄν τις πολλαπλάσιον ἑαυτοῦ αὔξων εἰς ἄπειρον· ἀλλ' οὐ διὰ τοῦτο ἔξω [τοῦ ἄστεός] τίς ἐστιν [ἢ] τοῦ τηλικούτου μεγέθους ὃ ἔχομεν, ὅτι νοεῖ τις, ἀλλ' ὅτι ἔστι· τοῦτο δὲ συμβέβηκεν.

way.[81] For Aristotle, there could not be such an actually infinite among numbers unless there were an actually infinite magnitude as well.[82] The same is also clear when one considers Aristotle's notion of the objects of mathematics. As we have seen, to study infinite numbers would be to study substance as having such a number, not as physical, but precisely insofar as there is a *so many*.[83]

If Cantor is right, there must first *be* an unending series of finite integers, else ω could not express "the idea that the entire assemblage (I) is given in its natural, orderly succession."[84] But if the unending series of finite integers exists, where is it? Is even the *first* number series truly infinite, let alone the transfinites? For the mathematician as such, the question is not an important one.[85] Yet Cantor has claims for transfinites that go beyond mathematics, even to the very limits of metaphysics. The answer to the question of where infinite numbers can be found which Cantor does ultimately supply—in the mind of God—is no adequate response to the Aristotelian objections. It is not in virtue of an intuition of the divine intellect that we are made aware of the unending series of finite integers—nor does Cantor claim this, to his credit. What does seem to be the case, rather, is that our everyday acquaintance with finite integers in counting, coupled with the mind's ability to add *ever one more*, raises the question of infinite numbers in the first place. On the other hand, historically, the controversy over infinite quantities had just begun.

We next turn to "the big three"—logicism, formalism, and intuitionism—briefly describing how each view addresses the question of infinity, the difficulties each encountered, and how such difficulties might be avoided via observations and distinctions arising from the mathematical philosophies of Aristotle and Thomas Aquinas.

[81] See especially *Physica*, III, 6, 206a18–24 and III, 7, 207b10–14.

[82] See Joseph S. Catalano, "Aristotle and Cantor: On the Mathematical Infinite," *Modern Schoolman* 46 (1968–69): 264–267.

[83] Although it appears, as we mentioned above, that one cannot rightly ask how many about an infinite number of things.

[84] Georg Cantor, "On Infinite, Linear Point-Manifolds," §11, 132.

[85] See Hilbert, "On the Infinite," 201. Though he claims the infinite does not exist in nature, nevertheless it is still mathematically legitimate to make a study of it.

The Big Three

The basic *logicist* position regarding mathematics is simple: math *is* logic. To the degree that logic itself is speculative, mathematics is speculative. The content of mathematics is the content of logic. Objects we have come to *distinguish* as specifically mathematical are merely abbreviations of more complex entities and expressions and can be reduced to them.[86]

> According to Frege's main thesis, arithmetic should be considered as a part of logic. The arithmetical notions—such as the notions of number, equality, sum—may be defined in purely logical terms; the theorems of arithmetic may be deduced from the principles of logic combined with the definitions of the arithmetical notions.[87]

Two distinct reductions, then, are required: that of mathematical notions to logical ones and that of mathematical propositions to logical ones.

Bertrand Russell who, along with Alfred N. Whitehead, followed Frege's lead in the logicizing of mathematics, made bold to define the science in the following manner:

> Pure Mathematics is the class of all propositions of the form '*p* implies *q*' where *p* and *q* are propositions containing one or more variables, the same in the two propositions, and neither *p* nor *q* contains any constants except logical constants. And logical constants are all notions definable in terms of the following: implication, the relation of a term to a class of which it is a member, the notion of *such that*, the notion of relation, and such further notions as may be involved in the general notion of propositions of the above form.[88]

The only concrete matter in the mathematical sciences—expressed by constants—is *logical*: all other notions are expressed by variables, whose content is indeterminate.

[86] We shall see an example of this shortly when we take up Russell's definition of number.

[87] Evert W. Beth, *The Foundations of Mathematics* (Amsterdam: North-Holland Publishing Company, 1959), 67–8.

[88] Bertrand Russell, *Principles of Mathematics* (Cambridge: Cambridge University Press, 1903), 3.

Russell's definition of number hinges upon a logical relation: in this case, that of a one-to-one correspondence, which relation implies another, *similarity*.

> Two classes are said to be 'similar' when there is a one-one relation which correlates the terms of the one class each with one term of the other class, in the same manner in which the relation of marriage correlates husbands and wives.[89]

Similarity is then applied to the problem at hand: the definition of number in purely logical terms.

> We may now go on to define numbers in general as any one of the bundles into which similarity collects classes. A number will be a set of classes such that any two are similar to each other, and none outside the set are similar to any inside the set. In other words, a number (in general) is any collection which is the number of one of its members; or more simply still: *A number is anything which is the number of some class.*[90]

A symbolic representation of a number (as Russell defines it, that is, purely in terms of logical constants alone) is exemplified in Rudolf Carnap's expression of the number two.

Let '$2_m(f)$' mean that at least two objects fall under the concept f. Then we can define this concept as follows…

$$2_m(f) = \mathrm{Df}(\exists x)(\exists y)[\sim(x = y) \cdot f(x) \cdot f(y)]$$

or in words: there is an x and there is a y such that x is not identical with y and f belongs to x and f belongs to y. . . Then we define the number two itself thus:

[89] Russell, *Introduction to Mathematical Philosophy*, 15–16. Russell's reference to the marital relation aids in the understanding of similarity. Assuming only monogamous marriages, one can say that the class of husbands is similar to that of wives without knowing—Russell's precise point—*how many* of each there are. According to Russell, one has avoided the use of counting in determining this similarity through the establishment of a one-to-one relationship between the respective members of each class (in this case, the relationship of monogamous marriage.)

[90] Russell, *Introduction to Mathematical Philosophy*, 18–19. Russell provides an argument that his last expression of the definition of number, though apparently circular, is in fact not so.

$$2(f) =_{\mathrm{Df}} 2_{\mathrm{m}}(f) \cdot \sim 3_{\mathrm{m}}(f)$$

or in words: at least two, but not at least three, objects fall under f.[91]

In general, the criticisms of logicism take two forms. First, logicization itself produces paradoxes. Secondly, to avoid the paradoxes, logicists (notably Russell himself) have had to adopt nonlogical axioms.

The most famous of the logical paradoxes is called the *Russell paradox*, since it was discovered by the man himself. In short, it hinges upon logicists' tendency to consider classes as objects—to substantialize classes.

Call that class the *Russell class* which contains all, and only, classes which are not members of themselves. Of course, it is quite reasonable to speak of classes which are not members of themselves—why would anyone speak of them otherwise? What would it mean to say that a class *is* a member of itself? Perhaps nothing. Still, the paradox arises, not with respect to whether one can have classes which *are* members of themselves, but with respect to whether classes can be considered members at all. Prior to deciding whether a given class is a member of the Russell class, one must determine whether it is a member of itself. The force of the paradox consists in recognizing that, because the notion of classes which *are* members of themselves seems odd and unlikely, then there must certainly be a Russell class: for there are surely classes which are *not* members of themselves. But *is* the Russell class a member of itself? Either it is, or it is not. If we assume that it is a member of itself, it follows that it *cannot* be a member of itself (a contradiction), for the Russell class contains only those classes which are *not* members of themselves. If we assume the opposite, that it is not a member of itself, then, since it contains all classes which are not members of themselves, it follows that it *must* be a member of itself (another contradiction). There seems to be no alternative, so the paradox presents as irresolvable.[92]

Russell sought to avoid the paradox by appealing to a principle whose basis in logic alone was at least questionable.

[91] Rudolf Carnap, "The Logicist Foundations of Mathematics" in Benacerraf and Putnam, *Philosophy of Mathematics: Selected Readings*, 42–3. Carnap is a later logicist.

[92] For an analysis of the *Russell paradox*, along with its symbolic representation, see Beth, *The Foundations of Mathematics*, 483.

> In Russell's opinion—which in this respect was shared by Poincaré—the paradoxes originate from a vicious circle. Consequently, he ensured their elimination by meticulously applying the so-called *vicious-circle principle*, which is stated as follows: "Whatever involves *all* of a collection must not be one of the collection."[93]

The *Russell paradox* bears upon the legitimacy of logicism itself to the degree that it takes classes to also be members—the substantialization of classes. As Russell's definition of number requires that a class be an object—ultimately, a number is a class of classes—the paradox runs to the very root of mathematics. Moreover, the root claim of logicism is compromised by means of the measures taken to avoid such contradictions—through the use of an hierarchical theory of types,[94] for example—undermining itself in the process.[95] Like paradoxes seem inherent to logicism's attempts to define the continuum in strictly logical terms.[96]

There is some dispute as to what *formalism* claims mathematics to be. According to Haskell Curry, formalism, much as its name suggests, considers mathematics as a study of systems of thought expressed in formal terms, i.e., apart from content.

> According to formalism the central concept in mathematics is that of a formal system. Such a system is defined by a set of conventions, which I shall call its *primitive frame*… The primitive frame specifies, independently of the representation, which elementary propositions are true, and therefore determines the meaning of the fundamental predicates. … The formalist definition of mathematics is then this: mathematics is the science of formal systems.[97]

[93] Beth, *The Foundations of Mathematics*, 497.

[94] A hierarchical theory of types is the concrete result of the application of the vicious-circle principle. It establishes a collection of classes in a definite order, such that a *lower* class (i.e., a class which is contained by another) cannot, at the same time, contain a *higher* class (i.e., a class which, mediately or immediately, contains it.)

[95] Apart from introducing extra-logical notions, the vicious-circle principle appears to be self-referentially inconsistent: for it refers to all principles which involve collections. Therefore, it itself cannot be a principle which refers to all principles which involve collections, for such principles constitute a class of propositions.

[96] These are taken up in Carnap, "The Logicist Foundations of Mathematics."

[97] Haskell B. Curry, "Remarks on the Definition and Nature of Mathematics," in Benacerraf and Putnam, *Philosophy of Mathematics: Selected Readings*, 203–4.

This understanding of formalist mathematics is echoed by one of its principal critics and the founder of *intuitionist* mathematics, L. E. J. Brouwer.

> For the formalist therefore mathematical exactness consists merely in the method of developing the series of relations and is independent of the significance one might want to give to the relations or the entities which they relate. And for the consistent formalist these meaningless series of relations to which mathematics is reduced have existence only when they have been represented in spoken or written language together with the mathematical-logical laws upon which their development depends, thus forming what is called symbolic logic.[98]

What is key here is the *lack of significance* assigned to the subject with which one is dealing.[99] Brouwer places emphasis upon the formalist's insistence upon *empty*—meaningless—concepts.

> The viewpoint of the formalist must lead to the conviction that if other symbolic formulas should be substituted for the ones that now represent the fundamental mathematical relations and the mathematical-logical laws, the absence of the sensation of delight, called 'consciousness of legitimacy,' which might be the result of such substitution would not in the least invalidate its mathematical exactness. To the philosopher or to the anthropologist, but not to the mathematician, belongs the task of investigating why certain systems of symbolic logic rather than others may be effectively projected upon nature.[100]

Meaning has so little to do with mathematics, says Brouwer, that formalists are altogether unconcerned whether the systems they have constructed coincide with reality or not; for the legitimacy of a formal system

[98] L. E. J. Brouwer, "Intuitionism and Formalism," in Benacerraf and Putnam, *Philosophy of Mathematics: Selected Readings*, 78–9.

[99] When, for example, we are wont to express a typical Aristotelian syllogism, not in terms of a real example (even if this be symbolized) but using variables, we are said to formalize the argument. 'Every a is b, every c is a, therefore every c is b' would be an example of such a formalization, where a, b, and c are taken to stand for anything whatsoever (and thus for no determinate thing.) This usage of symbols as variables may be contrasted to Aristotle's use of letters to stand for definite terms, as in the example he gives of a demonstration *quia*, at *Analytica Posteriora*, I 13 78a27–36.

[100] Brouwer, "Intuitionism and Formalism," 79.

is not in the least bit compromised by the representation which is subsequently given to it.

On the other hand, Kreisel is of the opinion that formalist mathematics is not so much concerned with formalization, as such, but with the reduction of those theorems in mathematics which would require an infinite number of steps to prove to finite proof procedures, to a *finitistic* method of demonstration.

> We note at once that there is no evidence in Hilbert's writings of the kind of formalist view suggested by Brouwer when he called Hilbert's approach 'formalism.' In particular, we could say that Hilbert wanted to eliminate the use of transfinite concepts from proofs of finitist assertions instead of referring to symbols and formulae as above. The symbols were a means of representation. The real opposition between Brouwer's and Hilbert's approach was not at all between formalism and intuitive mathematics, but between (i) the conception of what constitutes a foundation and (ii) between two informal ways of reasoning, namely finitist and intuitionist.[101]

Kriesel sees the difference between the schools in a shift of emphasis: from *the uninterpreted* and disregard for intuition to whether one will depend upon consistency or constructability as a criterion for truth.[102]

This confusion about formalism runs deep. Even the founder of the school, David Hilbert, seems to have espoused different mathematical methods, depending upon the subject under consideration.

> Does material logical deduction somehow deceive us or leave us in the lurch when we apply it to real things or events? No! Material logical deduction is indispensable. It deceives us only when we form arbitrary abstract definitions, especially those which involve infinitely many objects. In such cases we have illegitimately used material logical deduction, i.e., we have not paid sufficient attention to the preconditions necessary for its valid use.[103]

[101] Georg Kreisel, "Hilbert's Programme," in Benacerraf and Putnam, *Philosophy of Mathematics: Selected Readings*, 207.

[102] Of course, intuitionists require the actual constructability of a mathematical object in addition to finite proof procedures.

[103] Hilbert, "On the Infinite," 191–2. The translator's footnote regarding 'material logical deduction' reads as follows: "Throughout this paper the German word 'inhaltlich' has been translated by the words 'material' or 'materially' which are reserved for that purpose and which are used to refer to matter in the sense of the traditional distinction between matter or content and logical form.—Tr."

Hilbert seems not to have abandoned *references to content* in the mathematical sciences. In fact, he seems to regard content-based—that is, material—logic as an absolute requirement in some areas of mathematics. When we consider exactly what Hilbert means by "arbitrary abstract definitions, especially those which involve infinitely many objects" as well as what he means by mathematics as *applied* to the real world, however, what is most *distinctive* about formalism seems to be exactly what the name suggests: the necessity to regard certain propositions in a formal way. As to its application, that is markedly different from the study considered *as such*. On this point, Brouwer may be correct: an unconvincing application to the real world would not necessarily invalidate an otherwise valid formal system.

With respect to the question of *where* mathematical formalizations must occur, we turn to Hilbert himself for clarification.

> Even when from our finitary viewpoint a formula is valid with respect to what it signifies as, for example, the theorem that always
>
> $$a + b = b + a,$$
>
> where a and b stand for particular numerical symbols, nevertheless we prefer not to use this form of communication but to replace it instead by the formula
>
> $$a + b = b + a.$$[104]

The distinction Hilbert makes here is equivalent to one we made earlier—the difference between the utilization of a symbol to stand for a term with a *definite* sense and the utilization of a variable. At this point we wish simply to note Hilbert's preference for the latter—the formalization—over the former in cases where both are possible. Hilbert goes on to explain precisely what function the formula $a + b = b + a$ has in mathematics.

> This latter formula is in no wise an immediate communication of something signified but is rather a certain formal structure whose relation to the old finitary statements,
>
> $$2 + 3 = 3 + 2,$$
> $$5 + 7 = 7 + 5,$$

[104] Hilbert, "On the Infinite," 195–6.

> consists in the fact that, when *a* and *b* are replaced in the formula by the numerical symbols 2, 3, 5, 7, the individual finitary statements are thereby obtained, i.e., by a proof procedure, albeit a very simple one.[105]

Since one can arrive at the formulas themselves by way of *finitary* statements (such as 2 + 3 = 3 + 2) and return them back in a finite number of steps, it follows that formulas—*ideal* statements, in his parlance—can be utilized wherever finitary statements might be without violating the condition of using only finite proof procedures. One might ask, however, why one would *prefer* to use the formula for its corresponding finitary statements.

> A statement such as 'there exists' an object with a certain property in a finite totality conforms perfectly to our finitary approach. But a statement like 'either p + 1 or p + 2 or p + 3 … or (*ad infinitum*) … has a certain property' is itself an infinite logical product.[106]

In other words, *universal* judgments about numbers—for example, a + 1 = 1 + a—require that we need to determine whether this holds for 1, and 2, and 3, and 4, and so on, which reduces to a proposition with infinitely many parts, each of which must be true in order for the whole to be true. Why infinite parts? Because Hilbert regards the numbers as an infinite collection. A proposition in which a symbol stands for *any* number becomes a proposition about *all* numbers—'an infinite logical product.'

To avoid this, ideal statements will be required wherever we make universal judgments about numbers. *All even numbers are divisible by two without remainder* must first be expressed as an ideal statement in order that we might prove (or disprove) it in a finite number of steps. Hilbert's *material logical deduction* does not appear to *characterize* formalist mathematics, rather, if we expect to generalize, we will need ideal statements to do so. This final passage from Hilbert, noting the relation between symbolic logic and his own formalization of mathematics, summarizes the overall intention and characteristics of formalist mathematics quite well.

> It is consistent with our finitary viewpoint to deny any meaning to logical symbols, just as we denied meaning to mathematical symbols, and to declare that the formulas of the logical calculus are ideal statements which mean nothing in themselves. We possess in the logical calculus a symbolic language

[105] Ibid., 196.
[106] Ibid., 194.

> which can transfer mathematical statements into formulas and express logical deduction by means of formal procedures. In exact analogy to the transition from material number theory to formal algebra, we now treat signs and operation symbols of the logical calculus in abstraction from their meaning. Thus we finally obtain, instead of material mathematical knowledge which is communicated in ordinary language, just a set of formulas containing mathematical and logical symbols which are generated successively, according to determinate rules. Certain of the formulas correspond to mathematical axioms. The rules whereby the formulas are derived from one another correspond to material deduction. Material deduction is thus replaced by a formal procedure governed by rules.[107]

As to what mathematics *is*, Hilbert could not answer *episteme* according to its proper sense, for its notion necessarily references some definite objects about which one is theorizing—leaving aside, for the moment, the question whether such objects are real. Formalist mathematics lives up to its name. It generally ignores references to content and focuses upon logical form.[108]

In addition to the requirements of finite proof procedures and formalization in the case of universal propositions about numbers, Hilbert sets forth two additional conditions for mathematical legitimacy: consistency and completeness. As to the first, any system allowing for valid contradictions is of little value: if a contradiction can be proved, then anything can. Moreover, legitimate systems must be complete: every formalism that can be expressed must be decidable. How else could we be assured of (internal) consistency? Douglas Hofstadter summarizes Hilbert's so-called *program* with respect to these two conditions.

> Was it absolutely clear that contradictory results could *never* be derived, by any mathematicians whatsoever, following the methods of Russell and Whitehead? This question particularly bothered the distinguished German mathematician (and metamathematician) David Hilbert, who set before the world community of mathematicians (and metamathematicians) this

[107] Hilbert, "On the Infinite," 197.

[108] One might ask why we would take up the formalist position in considering the nature of arithmetic: for if the formalists regard mathematical propositions as statements without content, then what is to distinguish them as mathematical? Since formalism retains the notion of certainty in its insistence upon the rigor of proof and the certainty of conclusions, we consider it as having inquired into the science of arithmetic (and our own difficulty) to this extent. Furthermore, the formalists themselves claim to be doing arithmetic.

> challenge: to demonstrate rigorously—perhaps following the very methods outlined by Russell and Whitehead—that the system defined in *Principia Mathematica* was both *consistent* (contradiction-free), and *complete* (i.e., that every true statement of number theory could be derived within the framework drawn up in P.M.)[109]

We have noted, then, that both a symbolic form and completeness were conditions the formalist would require in mathematics. In the case of formalization, we have seen one difficulty looming: for if mathematics is to be divorced from definite content, how might one justify calling it a science at all? A further and—for the formalists—more telling difficulty concerns the question of completeness. Historically, formalism suffered its severest setback at the hands of Kurt Gödel, who proved that no adequate[110] formal system could possibly be complete and consistent, which is to say, no consistent formal system could decide all its validly expressible propositions.

We will not go into Gödel's incompleteness theorem in detail—let the following suffice. According to the proof, no adequate formal system could decide (prove true or false) a proposition equivalent to *this system is consistent*. Proving the internal consistency of all valid expressions within a system is impossible in principle.

This objection to formalist mathematics is—in short—fatal. To be unable to show that mathematics is even internally consistent is a serious problem for a formalism—that is, for a system which has *no other way* to indicate the *rightness* of its claims.[111] It is relevant also to a consideration of the *episteme* of mathematics. No one would describe as a science a system of reasoning whose consistency we could never hope to establish.

L. E. J. Brouwer founded *intuitionist* mathematics. As with the names given to *logicism* and *formalism*, the basis for this account of mathematics is *intuition*, a fundamental apprehension of the nature of primary mathematical objects. It is fair to say that it differs from formalism in supplying a *scientific* character to its study of things which are admittedly mathematical, while it differs from logicism in its consideration of mathematical

[109] Douglas R. Hofstadter, *Gödel, Escher, Bach: An Eternal Golden Braid*, (New York: Basic Books, 1979), 23–4.

[110] Adequate, that is, for purposes of mathematics.

[111] Even an incomplete material system can ground a particular claim in fact and so come closer to the truth. A formalist would always remain uncertain whether any of its claims is consistent with a future unexpressed claim.

objects *as such*. Brouwer's notion of mathematics, as he describes it here, is undeniably Kantian.

> However weak the position of intuitionism seemed to be after this period of mathematical development [the discovery of non-euclidean geometry,] it has recovered by abandoning Kant's *apriority* of space but adhering the more resolutely to the *apriority* of time.[112]

With a resolute *yes* in answer to the question *whether arithmetical statements are true,* Brouwer would go further to say, following Kant, that propositions arising from the intuition of time are *scientifically* true, are synthetic *a priori*. That *scientific via* Kant means something other than *episteme* becomes clear when we learn that intuitionists neither affirm nor deny the existence of separate mathematical objects.[113]

> I must still make one remark which is essential for a correct understanding of our intuitionist position: we do not attribute an existence independent of our thought, i.e., a transcendental existence, to the integers or to any other mathematical objects.[114]

Brouwer goes on to explain the principles of intuitionist mathematics.

> This neo-intuitionism [which has abandoned the spatial a priori but not the temporal one] considers the falling apart of moments of life into qualitatively different parts, to be reunited only while remaining separated by time, as the fundamental phenomenon of the human intellect, passing by abstraction from its emotional content into the fundamental phenomenon of mathematical thinking, the intuition of the bare two-oneness.[115]

Since the notion of unity itself is insufficient to produce a series, the intuition fundamental to mathematics must be of the *difference* within our apprehension of time. Through a process of abstraction we retain the

[112] Brouwer, "Intuitionism and Formalism," 80.

[113] *Separate*, not as *existing midway between the Forms and physical things*—as Plato would have it—but merely *existing separately from our thinking of them.*

[114] Arend Heyting, "The Intuitionist Foundations of Mathematics," in Benacerraf and Putnam, *Philosophy of Mathematics: Selected Readings*, 53.

[115] Brouwer, "Intuitionism and Formalism," 80.

notion of what is one (as continuous) yet many (as containing moments) which Brouwer calls the *bare two-oneness.*[116]

> This intuition of two-oneness, the basal intuition of mathematics, creates not only the numbers one and two, but also all finite ordinal numbers, inasmuch as one of the elements of the two-oneness may be thought of as a new two-oneness, which process may be repeated indefinitely; this gives rise still further to the smallest infinite ordinal number ω.[117]

Having explained these basal intuitions and the derivation of numbers from them, Brouwer sets a certain limit upon these activities, basing all construction proofs upon two operations: to construct a finite ordinal number, and to construct ω.[118]

Unlike formalism—where consistency substitutes for truth—intuitionists insist upon construction-proof in mathematics.

> For this reason [that all mathematical sets of units which are entitled to that name can be developed out of the basal intuition, and <that> this can be done only by combining a finite number of times the two operations <already given>] the intuitionist can never feel assured of the exactness of a mathematical theory by such guarantees as the proof of its being noncontradictory.[119]

For the intuitionist, every construction is a true mathematical object and every true mathematical object is either a construction or a basal intuition (which is an irreducible construction.)

At this point a prominent difficulty arises, for how could an intuitionist *disprove* a mathematical statement? In Euclid's *Elements*, for example, that

[116] This hearkens back to Plato's *same* and *other*.

[117] Ibid. It is interesting to note that the smallest infinite ordinal number appears to be arrived at through the repetition of a process an *indefinite* number of times.

[118] "All mathematical sets of units which are entitled to that name can be developed out of the basal intuition, and this can only be done by combining a finite number of times the two operations: 'to create a finite ordinal number' and 'to create the infinite ordinal number ω' … The intuitionist recognizes only the existence of denumerable sets, i.e., sets whose elements may be brought into one-to-one correspondence either with the elements of a finite ordinal number or with those of the infinite ordinal number ω." Brouwer, "Intuitionism and Formalism," 81. Note intuitionism's emphasis upon constructability as a foundation. It is not at all clear that ω is itself a finite construction.

[119] *Un*like formalism. Ibid.

there can be but a single line drawn perpendicular to a given line through a given point is shown by assuming the contradictory and demonstrating that such an assumption would entail an impossibility or contradiction. Does the intuitionist not want to establish, not only that ω *is the smallest infinite ordinal number* but also that *three is not equal to two*?

Taking an example of a geometrical theorem[120] we can see that, first, the intuitionist will have difficulty with all negative propositions and, second, demonstrating the truth of a contradictory may not always be possible. *A circle is not square* is shown to be true, not directly (for one could never construct such a thing,) but by supposing such a thing and deducing a contradiction from it. *A circle is square* can be shown to be false in the same manner.

> Consider the proposition 'A square circle cannot exist.' It is a proposition which Brouwer and Heyting admit as a theorem. … Brouwer describes it as a construction which consists in first supposing that we have constructed a square which is at the same time a circle, and then deriving a contradiction from the supposition.[121]

Two problems arise for intuitionist mathematics, then. The first looks to the proof of true negatives. How does such a process differ from the formalists' own method of establishing the legitimacy of mathematical objects, given the absence of contradiction? It seems either that such a process as is used by the intuitionist to prove a true negative proposition is fundamentally the same as that which is employed by the formalist (and, incidentally, by the logicist also) to establish the truth of mathematical propositions, or that this process is not a construction at all: either way, intuitionist mathematics is compromised.

Another, more serious, difficulty concerns the basal intuitions, specifically ω.

[120] Our choice of example looks to the supporting quotation which follows. An arithmetical example would have served as well, in principle.

[121] Korner, *The Philosophy of Mathematics: An Introduction*, 138. We might do well to note two implications of a strict construction-proof theory of mathematics. First, if, in certain cases, it is impossible to construct a mathematical entity and also impossible to disprove it, that is, to derive a contradiction from its supposed existence, then propositions about it will be neither true nor false: which is a denial of the Law of Excluded Middle. Further, as many mathematicians recognize the existence of certain mathematical entities which cannot be literally constructed (such as π) the *solvability of all mathematical problems* is also called into question by the intuitionist.

> A strict finitist would deny the (constructible) existence of infinitely proceeding sequences [like ω] in very much the same manner as the intuitionists deny the existence of actual infinities. *Infinitely* proceeding sequences, he might argue, unlike the *finitely* proceeding, outrun the human capacity for apprehension of the particular. We can imagine a process of stroke after stroke, up to a point; but there comes a point after which perception and intuition can no longer keep pace. To imagine 'in principle' that the process goes on without stopping, is no longer to imagine.[122]

Perhaps a problem arises here because we tend to consider Brouwer's construction operations (to create a finite ordinal number and to create the infinite ordinal number ω) as, in fact, only one. Perhaps, in seeing that an infinite ordinal differs in no essential way from a finite ordinal (excepting the number of members in each,) we regard the second operation as somehow *building* upon the first. In this case, the second operation would require an infinite number of steps, which is contrary to intuitionist principles, for they require (finite) constructions in mathematics. Thus, Brouwer's two basic operations must be (1) essentially distinct, and (2) each a *single* operation (as opposed to a series of operations.) If this were not the case, creating an infinite ordinal number would require an infinite number of steps.

But why would intuitionists admit an infinite act of construction—ω—in the first place? Presumably, following Brouwer, this follows immediately upon our apprehension of the primary intuition of time as a *continuum*. Yet according to what capacity are we able to intuit something admittedly infinite? The crux of the objection is this: if intuitionists deny the legitimacy of infinitely proceeding sequences on the grounds that they are infinite, why allow ω in one fell swoop, as it were? If we are mentally capable of intuiting ω, what prevents our mentally intuiting the cardinal number of all denumerably infinite ordinals[123]—of which ω is but the first?

It is possible that the mathematical philosophy of Aristotle and Thomas Aquinas can shed light on some of these difficulties—indeed, *any* legitimate explanation of mathematics would need to avoid paradoxes arising from a substantialization of classes, the incompleteness to which formal systems are prone as well as the difficulty in saying exactly *what* mathematics is about, as well as the apparent arbitrariness of intuition as an

[122] Ibid., 148.

[123] $\aleph_1$, a number which the intuitionists deny.

in-principle explanation and the difficulty which Kantian, *constructive* intuitions bring to proving true negatives. We hope to show not only that the Aristotelian and Thomistic account can avoid these difficulties, but also that it can supply an explanation of why they arose in the first place.

The single most fundamental notion in the logicists' reduction of mathematics to logic[124] is that of *class.* This concept is included as the genus of number itself and is at the basis of the *Russell paradox* and the difficulties encountered in logicists' attempts to avoid it. Is the *episteme* of mathematics—the cornerstone of the Aristotelian and Thomistic account of mathematics—subject to the same or an analogous difficulty?[125]

Number is *a plurality measured by the unit*, yet there is also a sense in which one number might *number* another—certainly it can *number* itself. Thus, six is numbered thrice by two, but only once by itself. For any given number, x, it is the case either that x numbers itself or that x does not number itself. Let a be the number of numbers which do not number themselves and b the number of numbers which do number themselves.[126] Either a numbers itself or it does not. Let us assume that it does not: hence, a will be *counted among* those numbers which do not number themselves; and yet a is the number of numbers which do not number themselves, and thus it numbers those numbers. But a is also one of the numbers that it numbers, and thus a numbers itself, which is contrary to the assumption. Let us now assume that a does number itself. Hence, a will not be *counted among* those numbers which do not number themselves,

[124] Of course this *reduction* is not—to the logicist—real, but only nominal. That is, what was thought to be distinct from logical notions (numbers) is shown to be definable in terms of logical notions. The logicist would not admit that numbers were anything more than classes, or perhaps properties of classes, themselves, although their being thought of (according to common opinion) as distinct requires the reduction.

[125] For a good discussion of paradoxes (with an emphasis on the *impredicable* paradox) from an Aristotelian perspective, see Henry Babcock Veatch, *Intentional Logic*, (New Haven: Yale University Press, 1952), 138–49. Veatch's answer to the paradoxes, however, differs from our own. A more general critique of logicist-type arguments and positions, again from an Aristotelian perspective, can be found in Charles DeKoninck, *The Hollow Universe* (Toronto: Oxford University Press, 1960).

[126] No claim is being made that a $\neq 0$. As will become evident, the paradox we are constructing here is based upon a fallacy of equivocation and is thus not a true paradox. It does seem likely, however, (even necessary in this case,) that the answer to the question 'how many numbers do not number themselves?' is 0. I have introduced this seeming paradox, an illustration of how the *Russell Paradox* might be translated into Aristotelian terms, in order to demonstrate a fundamental difference between the logicist notion of arithmetic and that proposed by Aristotle and Aquinas.

and thus, since *a* does not number those numbers which do number themselves, *a* will not number *a*. Which is again contrary to the assumption. It seems, then, that in either case we meet with a contradiction. Beyond arising from Bertrand Russell's notion of number as class, then, the *Russell paradox* also yields contradictory results when applied to *number* as defined by Aristotle and Thomas Aquinas. Yet the seeming parallel brings out precisely the distinction we wish to make between the notion of a *class* and that of a *nature*. For the paradox *just* presented is merely apparent—as will become clear.

When we speak of *a* as being the number of numbers which do not number themselves, we are considering that particular group which *a* numbers as a *class*. Indeed, predicating a number of some subject (*the numbers which do not number themselves are a in number*) is the clearest example we have of a subject which supposes *collectively* as opposed to *universally*. With collective supposition a logical subject is taken to refer, not to the individuals contained within it *individually*, as would be the case in the proposition *every dog is an animal*, but to the individuals contained within it taken *collectively*, as would be the case in the proposition *the legs of a spider are eight in number*. Their difference is clearly seen in the different *way* they predicate something of the subject. In the first case, the predicate, *animal*, is taken to apply to each and every dog, while in the second case, the predicate, *eight in number*, is taken to apply, not to *each* leg of the spider, but to the legs of a spider taken as a certain kind of whole—and not a *universal* whole. For the sense is not *this leg (of a spider) is eight in number* for each of the legs.

Just where the fallacy lies in our quasi-paradox is now apparent. In saying that *a* numbers those numbers which do not number themselves, we are regarding those numbers collectively—in no other way would we be able to say *how many* there are. Yet, in assuming that *a* either does or does not number itself, we are assuming that *a* either does or does not have a certain property in common with other numbers, which property (and its predication) has nothing to do with *how many* such numbers there are. When we say that, on the assumption that *a* is among those numbers which do not number themselves, the subject supposes *universally*—it pertains to each and every such number that it does not number itself. Yet—producing the paradox—when we say that the numbers which do not number themselves are *a* in number, the subject does not suppose universally. Which is to say it is not the same subject. It is not the case that *each* of the numbers which do not number themselves are *a* in number (or

are *numbered by a*,) and so it is not the case that *a*—included with them by assumption—is numbered by *a* because it is the *how many* of the group. A similar explanation holds for the assumption that *a* does number itself, with the consequent paradoxical result.

Classes are taken to include their members taken collectively, for in no case can the class itself be predicated of any of its members, even when we consider those classes which are members of themselves, though there might possibly be some class which is its sole member.[127] Thus, in no way can we assert that any dog *is* the class of dogs, or even that any particular class *is* the class of all classes—one of many members is not identical with its class.

In mathematics as *episteme*, however, numbers are understood as *natures*, not classes. *What it is to be the number eight*—the nature of eight—belongs to each and every instance of that number. Definitions are predicated of subjects supposing universally, not collectively. Nor is the nature of eight itself simply an expression of what it means *to be eight units*. As we saw earlier, the number eight is a *single thing*, or nature. It is not identical to an aggregate or collection of units. Were a number an aggregate, it is understandable that it *would* be defined as a class, for classes suppose collectively—indeed, the clearest *instance* of collective supposition is that involving number.

We have seen why our quasi-paradox is not real through an understanding of the differences between a nature and a class, which are based upon a difference in supposition. As we recall, Russell himself proposed to circumvent the paradox bearing his name through the application of the *vicious circle principle*, or, *that one cannot define something merely in terms of a whole to which it belongs*. This application produces a *hierarchy* of classes, or *types*—classes of individuals are of type one, classes of such classes are of type two, and so on. Even *classes* can be defined in this way by reference, not to a totality of *members* among which they are counted, but to a class which contains them but is not counted among them. On the other hand, the principle rendered the notion of class—previously regarded as intuitively simple and clear—somewhat muddled. For what has established this *hierarchy* among the classes other than Russell's desire to avoid the paradox? And why does logic alone require that one class *not* be defined in terms of a whole to which it belongs? Indeed, one author said, concerning the vicious-circle principle and the consequent theory of

[127] In this case any *predication* would be a mere statement of identity.

types, that "Russell surely was exaggerating when he claimed that the theory of types was inherently reasonable. On the contrary, its character is that of an arbitrary makeshift device for stopping the paradoxes."[128] Thus arises the claim that the logicist endeavor was preserved only by reference to nonlogical notions.

If the *episteme* of mathematics considers numbers as natures of a certain sort, it will avoid the *Russell paradoxes* (and possibly others). For it is the *collective* supposition numbers as classes bring to mathematics which engenders the paradox (it *defines* a number *as* the collective whole of the many members it contains, including itself). If the distinction between the number eight as 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 and the number eight as a certain *nature* is maintained, as evidencing the distinction between the *techne* and the *episteme* of arithmetic, respectively, paradoxes like Russell's can be avoided.[129] His logical *reduction* of the number two to the class of twins reduces mathematics to its art, and excludes the *episteme*, for it considers two simply and entirely as made up of actually distinct parts. For Russell, two *is* one plus one.

As we saw, formalism discovered its fatal flaw *via* Kurt Gödel.

> Gödel's way was to argue directly to the incompletability of elementary number theory, by showing how, for any given proof procedure P for elementary number theory, a statement S_p of elementary number theory can be constructed which will be true if and only if it is not provable by the procedure P. Either S_p is provable, in which case it is false and so the general proof procedure P is discredited, or else S_p is true and not provable, in which case the proof procedure P is incomplete.[130]

[128] Stephen F. Barker, *Philosophy of Mathematics* (New Jersey: Prentice-Hall, Inc., 1964), 87–88. We are not claiming here that the vicious-circle principle is untrue, only that it appears to compromise the reduction of arithmetic to logic, particularly to the notion of a *class*—to which we have opposed the notion of a *nature*—which, we maintain, is the source of these antinomies.

[129] Cf. our discussion of the art and *episteme* of mathematics in Chap. 6.

[130] Willard Van Orman Quine, *Methods of Logic* (New York: Henry Holt and Company, Inc., 1959), 246. So, too, Hofstadter: "In the thirty-first year, however, Gödel published his paper, which in some ways utterly demolished Hilbert's program. This paper revealed not only that there were irreparable 'holes' in the axiomatic system proposed by Russell and Whitehead, but more generally, that no axiomatic system whatsoever could produce all number-theoretical truths, unless it were an inconsistent system! And finally, the hope of proving the consistency of a system such as that presented in P.M. was shown to be vain: if such a proof could be found using only methods inside P.M., then—and this is one of the most mystifying consequences of Gödel's work—P.M. itself would be inconsistent!" Hofstadter, *Gödel, Escher, Bach: An Eternal Golden Braid*, 24.

Though undeniably a problem for Hilbert's formalism, is a real problem otherwise? Is it also applicable to the *episteme* of mathematics, for example?

There *is* a real problem here and one could quite reasonably expect Gödel's finding to apply to *any* systematic account of mathematics. On the other hand, the very *difference* we have highlighted between mathematics as *episteme* and formalism may very well render the former immune to Gödel's discovery.

The problem with incompleteness, for the formalists, is that any proposition expressible in their system *is a mathematical object*. In other words, in formalizing mathematics—reducing it to a study of symbols without content[131]—they have reduced *constructive proof* (corresponding in logicism to the *reduction* of the notion of number, say, to that of a certain class of classes, and in intuitionism to positive or negative construction) to the mere *expressibility* of a proposition in the symbols and syntax of a given system. Mere expressibility is the *palest* of constructive methods. A proper understanding of Gödel's discovery requires that mathematical propositions be symbolically expressed—thereby *becoming* mathematical objects themselves.

The *episteme* of mathematics as described is analogous to this notion of a formalized system in some respects. For example, any number which can be constructed from the unit and whose assumption does not entail a contradiction *is* a property of the unit and is thereby *real*. Hence the *problem* of completeness. For such numerical constructions correspond to symbolically expressible claims, while establishing noncontradiction and deducibility from the unit corresponds to formalist *meta-mathematics*—the establishment of noncontradiction in a given system. In this way, then, the Aristotelian and Thomistic account of mathematics may be found subject to Gödelian undecidability.

Upon closer inspection of the difficulty, however, undecidability fails to apply. As we just mentioned, were statements made *about* a given formalized system not expressible in terms of the system itself, Gödel's conclusion would not follow. He himself, in his proof, assigns to each symbol,

[131] We note here, once again, that formalists do not generally maintain that the symbols did not *originate* within a material context—that content is implied—but rather that they are *considered* without content, in a sort of abstraction—that they are *uninterpreted*.

formula, and deduction found in the system[132] a unique number, which, by assumption, is also expressible in terms of the system: for, as we saw for Russell, number itself is definable wholly in terms of logical notions.

Establishing consistency in a formal system requires showing that a contradiction *could not result*. Gödel's proof shows, not that an *arithmetical* contradiction (e.g., that 1 = 2) results, but that a *meta-mathematical* proposition—a symbolically expressed claim that the system is inconsistent—does. A *complete* system—where even *such* a proposition is decided—is obviously inconsistent, whereas a system leaving such a proposition undecided is obviously incomplete—by at least one.

Would Aristotle and Thomas Aquinas see *meta-mathematical* statements—claims made *about* mathematics—as being mathematical claims themselves? Is *the episteme of mathematics is consistent* expressible in mathematical terms?

It seems impossible. For arithmetic and geometry have determinate *subjects*. They are not groups of content-less, or—in fairness to the formalists—uninterpreted, sentences. Indeed, the content itself of mathematics—continuous quantity in three dimensions and the mathematical unit—is taken from *outside* the mind not merely *syntactically arranged*, as are Hilbert's *ideal statements* and symbols. It is precisely *because* symbols are considered formally that *meta-mathematical* statements *themselves* can be formalized. After all, taking *a* to stand for a given formal, syntactically correct expression, *a* is *3 is the first prime number* and *this system is consistent* indifferently.

> The reasoning in the Richard Paradox is evidently fallacious. Its construction nevertheless suggests that it might be possible to 'map' (or 'mirror') meta-mathematical statements *about* a sufficiently comprehensive formal system *into* the system itself. If this were possible, then meta-mathematical statements about a system would be *represented* by statements within the system. Thereby one could achieve the desirable end of getting the formal system to speak about itself—a most valuable form of self-consciousness.[133]

[132] Gödel takes as his own example the formalization found in *Principia Mathematica*, although any formalized system which can express addition, multiplication, and quantification with respect to the natural numbers suffices equally well.

[133] Ernest Nagel and James R. Newman, "Gödel's Proof," in *The World of Mathematics*, v. 3 (New York: Simon & Schuster, 1956), 1687. Emphasis mine. The Richard Paradox results from considering statements about a system as being expressible within a system. Our point here is that the incompleteness proof avoids this paradox (and the fallacy it entails) by considering only sufficiently formalized systems.

The formalization which gives this mathematical view its name is a necessary condition for expressing *meta-mathematical* statements in a mathematical system—the very heart of the incompleteness proof of Kurt Gödel.

Claims made about continuous quantities, numbers, and the unit, as such, are not claims about their logical relations: for these are neither numbers nor the arithmetical unit.[134] It is precisely in requiring a definite subject of the *episteme* of mathematics that we avoid the incompleteness or inconsistency of formalized systems whose completeness and consistency David Hilbert and the formalist school sought to establish.

One distinction remains to be made between formalist mathematics and that of Aristotle and Thomas Aquinas, one central to understanding the difference between *episteme* and Hilbert's *material* arithmetic. It also serves to illustrate one source of the difficulties inherent in formalism.

As we saw with Hilbert, a universal mathematical judgment made about the numbers entails an infinite logical statement, hence the need for corresponding finite—albeit formalized—mathematical claims. Hilbert's argument for such a requirement could not be better for our purposes, for he begins with a discussion of a Euclidean proposition about prime numbers.

> By a well-known method due to Euclid we can give a proof, *one which remains entirely within our finitary framework*, of the statement that between $\mathfrak{p} + 1$ and $\mathfrak{p}! + 1$ there exists at least one new prime number. The statement itself conforms perfectly to our finitary approach, for the expression 'there exists' serves only to abbreviate the expression: it is certain that $\mathfrak{p} + 1$ or $\mathfrak{p} + 2$ or $\mathfrak{p} + 3$ . . . or $p! + 1$ is a prime number. … Although this theorem is a much weaker statement in terms of content—it asserts only part of what the euclidean theorem asserts—and although the passage from the euclidean theorem to this one seems quite harmless, that passage nonetheless involves a leap into the transfinite when the partial statement is taken out of context and regarded as an independent statement.[135]

Why such a leap? Because the *Euclidean* theorem is a *universal* judgment about an infinite totality, namely, *the numbers*.

[134] For a good discussion of the Richard Paradox (engendered by the failure to make this distinction) and of the logical fallacy it entails, see Nagel and Newman, "Gödel's Proof," 1686–7.

[135] Hilbert, "On the Infinite," 193.

> A statement such as 'there exists' an object with a certain property in a finite totality conforms perfectly to our finitary approach. But a statement like 'either $+ 1$ or $\mathfrak{p} + 2$ or $\mathfrak{p} + 3$ … or (ad infinitum) … has a certain property' is itself an infinite logical product. Such an extension into the infinite is, unless further explanation and precautions are forthcoming, no more permissible than the extension from finite to infinite products in calculus. Such extensions, accordingly, usually lapse into meaninglessness.[136]

Because Hilbert takes the Euclidean theorem to mean that a prime will be found between $\mathfrak{p}$ and $\mathfrak{p}! + 1$—where $\mathfrak{p}$ is regarded as true for *all prime numbers*—we will inevitably be confronted with an infinite disjunctive proposition.[137]

Hilbert is adamant about the need to excise infinite processes from mathematics—to the point of sarcasm.

> We find writers insisting, as though it were a restrictive condition, that in rigorous mathematics only a *finite* number of deductions are admissible in a proof—as if anyone had succeeded in making an infinite number of them.[138]

Barring such infinities, then, Hilbert reduces universal claims to *ideal statements*—which finite formalizations *are* the object of mathematical study. Even if *interpreting* $a + 1 = 1 + a$ entails an infinite proof procedure, the formula itself does not.

> We regard expressions containing letters as independent structures which formalize the *material* theorems of number theory. In place of statements about numerical symbols, we have formulas which are themselves the concrete objects of intuitive study. In place of number-theoretic material proof, we have the derivation of a formula from another formula according to determinate rules.[139]

This limitation imposed upon mathematics is extreme. *Material* mathematics is only about *individual* numbers and universal claims about numbers are reduced to formalizations, which thereby become mathematical objects.

[136] Ibid., 194.

[137] We shall consider this Euclidean proposition in detail later, in our discussion of intuitionist mathematics.

[138] Ibid., 184.

[139] Ibid. Emphasis mine.

Most importantly, properly universal mathematical claims are undecidable *precisely because* they are thought to have an infinite *signification*.[140]

As we have seen, Aristotle and Thomas Aquinas approve of mathematical infinities no more than does David Hilbert. Would *they* exclude universal claims—for example, the *no greatest prime* theorem—from the *episteme* of mathematics?

Of course, they do not—indeed, *episteme* is all *about* the universal and the necessary, as we have seen. On the other hand, unlike Hilbert, they do not regard *the numbers* as an infinite totality in the first place. More the contrary—it is in any consideration a *finite* collection, to which one may add as many numbers as one chooses. In other words, in accordance with the distinction we made above, the logical subject of *no prime number is the greatest prime number*, supposes *universally*, not *collectively*. Thus, one can demonstrate—albeit indirectly—that the *nature* of prime number is incompatible with the *nature* of being quantitatively *greatest*.

To take an example instancing how Aristotle and Thomas Aquinas see things, the universal affirmative proposition *every number is measured by the unit* is, for them, a self-evident *principle*, since it merely predicates the definition of that which is defined by it: a necessary condition of every definition. Even so, Hilbert would find the *mathematical* claim problematic. For though the claim might be formally expressed in this way—

$$(a)\left[N(a) \rightarrow M(a)\right]$$

which expression he would easily grant—it is *materially* subject to the same concern he gave above, that is, it implies an infinite number of claims.

> For example, the statement that if *a* is a numerical symbol, then *a* + 1 = 1 + *a* is universally true, is from our finitary perspective *incapable of negation*. We will see this better if we consider that this statement cannot be interpreted as a conjunction of infinitely many numerical equations by means of 'and' but only as a hypothetical judgment which asserts something for the case when a numerical symbol is given. From our finitary viewpoint, there-

[140] Thus Hilbert: "Consider one formula out of the rich variety of elementary formulas of number theory, e.g., the formula

$12 + 22 + 32\ldots + n2 = (1/6)n(n + 1)(2n + 1)$

Since we may substitute any integer whatsoever for *n*, e.g., *n* = 2 or *n* = 5, this formula *implicitly contains infinitely many propositions.* This characteristic is essential to a formula." Ibid., 186–7. Emphasis mine.

> fore, we cannot argue that an equation like the one just given, where an arbitrary numerical symbol occurs, either holds for every symbol or is disproved by a counter example. Such an argument, being an application of the law of excluded middle, rests on the presupposition that the statement of the universal validity of such an equation be capable of negation.[141]

This is, perhaps, the most important portion of Hilbert's whole argument. In essence, he is claiming that, if $a + 1 = 1 + a$ is to hold universally, that is, as a theorem of arithmetic, it is necessary that its opposite be capable of negation, otherwise we could not apply the law of excluded middle—a necessary condition for any proof which purports to establish the truth of *one* proposition and the falsity of its *opposite*. In other words, in order to prove that *either* $a + 1 = 1 + a$ *holds for every* a *or there is some* a *for which* $a + 1 \neq 1 + a$, we must first establish that the opposite of this universal judgement can be expressed in finite terms.

If we are to resort to finite proof procedures *alone*, says Hilbert, we cannot regard the universal claim that $a + 1 = 1 + a$ holds as an infinite conjunction—$(1 + 1 = 1 + 1) \cdot (2 + 1 = 1 + 2) \cdot \ldots$.—but only as a hypothetical proposition testing whether the rule applies to individual instances or not. One cannot be certain that the negation of this judgment is expressible in *finitistic* terms, for it may follow that some a cannot itself be so expressed. Thus, according to Hilbert, the impossibility of negating $a + 1 = 1 + a$ if it is taken as referring to numbers makes the statement itself meaningless. Our only recourse is to render such universal claims as formulas and resort to an intuitive study of *them* as the objects of mathematics.

Returning to the Aristotelian and Thomistic claim that *every number is measured by the unit*. We have already noted the importance of differentiating a *universally* supposing subject from one which supposes *collectively*. But does the *every* in the claim commit us to an infinite numerical universe? Not at all. The statement maintains that the *nature* of number—which, in fact, belongs to anything which is a number, *howsoever many of these you might wish to consider*—suffices for the property of being measured by the unit. One might discern a difference between saying *every number is measured by the unit* and *all numbers are measured by the unit*; in the latter case, one might take the subject to suppose collectively, not universally, as if we were speaking, not about the nature of number, but about a group or set of objects. Thus, following Aristotle and Thomas Aquinas,

[141] Hilbert, "On the Infinite," 194.

we know with complete confidence that *there is a number greater than every number*, but not that *there is a number greater than all numbers.*

$$a+1=1+a,$$

if taken universally and as signifying something about *numbers*, presents difficulties for David Hilbert, as we have seen. Not so in mathematics as *episteme*, since it can be traced back to the nature of number, the nature of the process of addition, the nature of equality, and the nature of the unit. Since we do not regard *the numbers* as an infinite totality, but only the four natures specified, there is no problem with an infinite disjunctive proposition: in fact, it is only *because* Hilbert regards the numbers as an infinite totality that an infinite disjunction arose. *Disprovability* in the *episteme* of mathematics is always feasible—there are only a finite number of natures relevant to any particular consideration.

As to intuitionism, we have already noted that they need not deny the Law of Excluded Middle—even as selectively as they propose—to defend the place of constructability in mathematics.[142] Indeed, Aristotle's solution to the Meno dilemma and Thomas Aquinas' development of things virtually known have their place in addressing the very mathematical *completeness* to which formalism aspires.

Here, however, we wish to return to the acts whereby this school of thought acquires its name: intuition itself. The first point we should make is that mathematics as *episteme* is not founded upon *intuition,* as understood by the intuitionists. First, mathematics does have, in principle, a basis in reality—in the sensible world—even if we consider it in abstraction from that world. Secondly, the (subjective) structure of our apprehension of time does not found *episteme* at all, mathematical or otherwise. At issue also in this latter case is the fundamentally skeptical distinction between *phenomena* and *noumena*.[143]

How could our intuiting an infinite ordinal, ω, be anything more than a problem peculiar to the intuitionists? It *is* a problem for them in that they readily assume an infinite whole in a single act of apprehension. On the other hand, it is a problem that underlies *all* attempts at founding

[142] See, especially, Chap. 7.

[143] On the other hand, among the three schools, intuitionism is most like the mathematical realism of Aristotle and Thomas Aquinas, in that it deduces the whole from the apprehension of a principal subject.

mathematics in human apprehension—be that continuous quantity in three dimensions, the unit of mathematics, basal two-oneness, or ω.

Of more direct consequence is intuitionism's inclusion of *indirect, non-constructible proofs* of true negatives. We shall consider this latter difficulty first.

As we have seen, the intuitionists maintain that nothing can be considered a valid mathematical object unless it can be finitely constructed—hence their exclusion of $\aleph_1$. As ω, the least of the infinite ordinals, is known in principle, it might seem that other infinite ordinals would be known. The problem is that such ordinals (not being themselves intuited) would require an infinite construction process—*constructed* infinities cannot be made in a finite number of steps.[144]

On the other hand, what sort of construction does occur when we reject a mathematical proposition whose subject cannot be constructed, for example, *an even odd number cannot be measured by the unit alone*? The proposition is clearly true, yet the subject cannot be constructed, and thus it would seem impossible to show that the predicate *is truly separated from* the subject. The intuitionists, however, allow construction in this case, *not* of the even odd number, but of *the proof itself*—deducing a contradiction from the *supposition* that an even odd number has been constructed.

This is not *construction* in the same sense as *constructing* the rational number 4/5 or the whole number 10^{10}. In fact—as we noted just above—such *indirect* proofs are part and parcel of the methods of *any* of the mathematical schools of thought.

A similar argument might be made against the mathematics of Aristotle and Thomas Aquinas, who require that any number must be derivable from the unit, for example, without contradicting its nature—itself a sort of construction. How would *they* deal with the proposition *an even odd number is not measured by the unit alone*?

Clearly, we cannot derive this number from the unit without contradiction. How, then, are we to *establish* the separation of predicate from subject? In fact, however, the same problem arises with propositions like *no prime number is the greatest prime number*. How can one establish its (undeniable) truth absent a construction of that prime number about which it is necessarily false to say that it is greatest. In other words—once again—how can such a construction *not* entail an infinite procedure? After

[144] Of course, ω itself might seem to be an exception to this, but it is also a principle and not *merely* a construct.

all, we are able to construct any prime number we choose and establish that *it* is not the greatest by way of constructing a greater. Yet—as Hilbert would be happy to point out—we cannot establish a universal proposition upon the basis of such an enumeration.

Arguments whereby propositions are established in this way are *reduction arguments*[145] or *indirect proofs*. In fact, an excellent example of such is Euclid's own *no greatest prime* theorem.[146]

Suppose a certain number of primes, *A*, *B*, and *C*. It is required to show that there are more prime numbers than any given number.[147] Let the least number measured by *A*, *B*, and *C* be taken, and let us call it *D*. Then let *E* be equal to *D* + 1. Either *E* is prime or it is not. Let us suppose it is prime. Then there are more prime numbers than the given number. Let us suppose it is not prime. We already know (from a previous proposition) that any composite number is measured by some prime number. Let *E* be measured by the prime number *G*. Now *G* is not equal to either *A*, *B*, or *C*. (At this point, the argument employs a *reductio ad absurdum*.) For, if it is possible, let it be so. Since *A*, *B*, and *C* measure *D*, then (by assumption) *G* will also measure *D*. But it is given that it measures *E*, and therefore it will also measure the difference between *D* and *E*. But this difference is a unit, and thus a number will measure the unit: which is absurd. Therefore, the prime number *G* is not equal to *A*, *B*, or *C*. Therefore, there are more prime numbers than *A*, *B*, and *C*.

Focusing upon the latter part of the proof—where *E* is either prime or not—suppose a concrete instance, the given primes being 2, 5, 7, and 11.[148] The least number measured by these primes is 770, to which we add the unit, with a result of 771. 771 is not itself prime, for it is divisible by 3. On the supposition of what is not true—that there are no more primes than 2, 5, 7, and 11—we can establish the opposite, that there must be an additional prime

[145] Different names are applied to these arguments, as *reductio ad impossibile*, *reductio ad absurdum*, and *reductio ad contradictionem*. They are instances of the general form *modus tollens*.

[146] Euclid, *The Elements*, IX, 20.

[147] That is, that *no prime number is the greatest prime*, since all prime numbers known at the time of proof could be the given primes, in which case any additional prime would be greater than the greatest given.

[148] Our choice of numbers is deliberate, for in a simpler example, such as 2, 3, and 5, the composite number + 1 is already equal to a prime number: hence the example would illustrate the point only by establishing the simplest case.

770 is divisible into its factors and, since it is not prime, 770 must be measured by a prime. But—by supposition—only 2, 5, 7, and 11 are prime. Therefore, both 770 and 771 will be measured by, say, 2. It follows that 2 will also measure the difference between 770 and 771, which is the unit. But the unit is *per se* indivisible. Therefore the supposition that 2, 5, 7, and 11 are the only primes is demonstrably false. (The proof is substantially the same for each of the four possible alternatives.) But why should the assumption of something we know—or at least suspect—to be false lead us to truth?

The answer lies in the very nature of reasoning. False conclusions can arise only when our reasoning is faulty or at least one premise is false. Establishing validity is (largely) independent of meaning—let us assume that our reasoning is correct. Assuming this, we can only arrive at a false conclusion through (at least one) false premise. But the conclusion of the reduction is decidedly false—independently of the argument in which it is found.[149] Thus, the only way one could be validly led to this conclusion would be through a false premise. Since, in a reduction, we assume that what we intend to disprove is true, and since this then leads to a false conclusion, a check of the truth of the alternate premise of the argument indicates what we had supposed all along: that the premise initially assumed contrary-to-fact must be the false premise. In this way, a reduction establishes, not the existence or nonexistence of a given *object*, but the truth or falsity of a *proposition*. We do not make an *object* but a true or false judgment, and so our making is of a different order altogether.[150]

It turns out that the very *idea* of *construction* is manifold. It can mean the actual production (by counting, or other such operations) of an object we would call a number, or it can refer to the production of a proposition—after all, logic, too, is among the liberal arts.

Finally, what of intuitionists' specific—basal—intuitions? Why *these* principles? Recall that it is in light of assumed principles that certain notions are or are not included within mathematics. Intuitionists exclude *certain ones*, such as the non-denumerables and anything which cannot be derived from the *two-oneness* or ω. In fact, Aristotle and Thomas Aquinas are even *more* restrictive, disallowing negatives, fractions, and even zero. After all, they say, such notions are incompatible with the unit, either as a

[149] It is absurd, a contradiction, impossible, or something of this sort. In any case, patently false.

[150] For more on being as divided into the categories as against the being of truth, see *Metaphysica*, V 7 especially 1017a31–34.

principle or as indivisible. But—one might ask—why restrict the notion of the unit to this? Similarly, against the intuitionists, if one admittedly infinite number is allowed, namely, ω, why not others? By what criteria are certain intuitions excluded and others allowed?

Aristotle and Thomas Aquinas claim that mathematics is a *science* in the strict sense of that word. The answer to the present difficulty lies in what we take as the real basis for the truth of mathematical claims. In their case, it is continuous quantity in three dimensions and the mathematical unit. The unit is regarded as a measure and indivisible—they argue— because that is the nature it has in reality, of which nature we become aware through a process of formal abstraction. The intuitionists, for their part, claim that their principles arise immediately from sense apprehension through the *a priori* form of time. To be quite honest, this question applies to the other mathematical schools of thought—indeed, to *any* claim to a philosophy of mathematics—as well. Why regard logical principles as immediately known and thereby a sufficient basis of all mathematics? Why consider logical form alone as a fitting object of mathematical study?

The answer to this last question is beyond our capacity to formulate precisely. Obviously, a difference in principle supplies no common ground from which to proceed when arguing about principles. To the formalists' objection that intuitionism unnecessarily restricts the field of valid mathematical endeavors, the intuitionist replies that the restriction is not unnecessary, but only as necessary as the nature of the subject warrants.

There may be some progress to be made, even so. Peculiar to this form of *episteme* is an included account of activities which fall outside it—considered as such—yet pertain to and aid it in the completion of its work: mathematics as an art. Here one finds a truth and falsity proper only to art, along with the *arbitrariness* associated with the activities of that capacity. One can surely make *i*—one can think it, its relation to the natural numbers, its name, and its symbol. On the other hand, such a (mere) assumption is inadequate to the requirements and purposes of *episteme*.

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CHAPTER 12

Mathematical Realism and Anti-Realism

The Controversy

We have seen recent, famous attempts at establishing mathematics on a secure foundation—indeed, the intense activity among mathematicians and philosophers of mathematics in the early part of the last century is often referred to as *The Crisis in the Foundations of Mathematics*.[1] There is one distinction among the views to which we have given relatively lesser attention—whether logicists, formalists, and intuitionists are or are not *realists*. The *question* of mathematical realism vs. mathematical anti-realism, of course, has occupied us throughout—this book is *about* the mathematical realism of Thomas Aquinas, after all. Still, we have most recently focused upon the *epistemological* aspect of recent mathematical theories and not the ontology of mathematical objects. It is that topic with which we are directly concerned in this chapter.

The *distinction* between mathematical realism and mathematical anti-realism is, of course, an ancient one. We need only recall our brief, initial review of Plato's mathematical views—and Plato's account is certainly not without its opponents.[2] Moreover, *realism* and *anti-realism* extend to

[1] Cf. the article of that name by José Ferreirós in *The Princeton Companion to Mathematics*, eds. Timothy Gowers, June Barrow-Green, and Imre Leader (Princeton: Princeton University Press, 2009), 142–56.

[2] Some argue that Aristotle himself is a mathematical anti-realist, for example, though, as we have seen, he is a realist, albeit not of a Platonic sort.

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areas beyond mathematics. Indeed, anti-realism makes its first appearances in natural philosophy—quite probably through the reductionism of Democritus or others like him.[3]

In the mathematical realm it has received recent and widespread attention *via* a 1973 article by Paul Benacerraf entitled *Mathematical Truth*.[4] In sum—and as Stewart Shapiro presents it—the question is not as simple as a simple *realism* vs. an *anti-realism*, but *realism* vs. *anti-realism in ontology* as well as *realism* vs. *anti-realism in truth-value*.[5] As with the Kantian analyses of analytic and *a priori* vs. synthetic and *a posteriori*, there are other possibilities—though one does *expect* a close mapping between the two.

> There is a prima facie alliance between realism in truth-value and realism in ontology. … The ontological thesis that numbers exist objectively may not directly follow from the semantic thesis of truth-value realism. There may be objective truths about mind-dependent entities. However, the objective existence of mathematical objects is at least suggested by the objective truth of mathematical assertions.…
>
> There is another close alliance between what I call idealism in ontology and idealism in truth-value. The former contends that numbers, for example, are dependent on the human mind. This at least suggests that mathematical truth is also dependent on the human mind. The same goes for the other sorts of anti-realisms. Whatever one says about numbers at least suggests something similar about mathematical truth.[6]

As to *ontology*, realisms and their corresponding anti-realisms are *metaphysical* in nature. Aristotle's denial of Plato's mathematical *intermediates* (on the grounds, for example, that it is unclear what access sentient animals might have to such immaterial realities or on the basis of a modified *third man* argument) is metaphysical in nature. At issue is not how we might give an account of the mathematical knowledge we have obtained—an *epistemological* question—but whether our knowledge is grounded in realities that exist independently of our thinking of them.

Benacerraf poses his question at the juncture of these two realms.

[3] Democritus famously denies the external reality of sensible characteristics like color and flavor, reducing them *in re* to the shapes, arrangements, and positions of atoms.

[4] As Stewart Shapiro notes, this article "…continues to dominate contemporary discussion in the philosophy of mathematics." Stewart Shapiro, *Thinking About Mathematics* (Oxford: Oxford University Press, 2000), 31.

[5] Ibid., 25–33.

[6] Ibid., 31–2.

> Accounts of truth that treat mathematical and non-mathematical discourse in relevantly similar ways do so at the cost of leaving it unintelligible how we can have any mathematical knowledge whatsoever; whereas those which attribute to mathematical propositions the kinds of truth conditions we can clearly know to obtain, do so at the expense of failing to connect these conditions with any analysis of the sentences which shows how the assigned conditions are conditions of their truth.[7]

As Shapiro, Hart,[8] and others note, Benacerraf's observation revisits the ancient question of *access to abstract mathematicals* vs. *a priori truth-conditions*.

This part of the book is concerned with a brief consideration of recent mathematical realisms and anti-realisms—and where among them the realist mathematical philosophy of Aristotle and Thomas Aquinas might be found.[9]

Anti-Realisms

As to the anti-realists, we have already looked to the tradition: intuitionism itself. Small wonder that expressly adopting Kant's intuition of time as one's mathematical foundation leads to *ontological anti-realism*.[10] Moreover, where could an insistence upon constructability—even in the face of a denial of the law of excluded middle—lead but to mind-dependent objects? This sort of anti-realism—in ontology and truth-value alike—is a reasonable consequence of the synthetic *a priori*.

[7] Benacerraf, "Mathematical Truth," 403.

[8] Cf. W. D. Hart, "Benacerraf's Dilemma," *Crítica: Revista Hispanoamericana de Filosofía*, 23, no. 68 (August 1991): 87–103.

[9] In *Thinking About Mathematics*, Shapiro supplies a helpful summary of prominent contemporary views, arranged along realist and anti-realist lines. "In recent literature on philosophy of mathematics, Gödel (1944, 1964), Penelope Maddy (1990), Michael Resnick (1997), and myself (Shapiro 1997) are thoroughgoing realists, holding both realism in ontology and realism in truth-value. … On the contemporary scene Hartry Field (1980), Michael Dummett (1973, 1977), and the traditional intuitionists L. E. J. Brouwer and Arend Heyting are thorough-going anti-realists, concerning both ontology and truth-value. Field holds that mathematical objects do not exist and that mathematical propositions have only vacuous truth values." Shapiro, *Thinking About Mathematics*, 32. For our purposes in this chapter, we will focus upon Hartry Field's *anti-realism*, as well as Quine-Putnam indispensability and Shapiro's own *structuralist realism*.

[10] To be fair, Kant does not insist that space and time are not real, since he only concludes that we do not—cannot—know of their extra-mental reality, space and time being preconditions for sensory experience itself.

> One version of truth-value anti-realism is that unambiguous mathematical statements get their truth-values in virtue of the human mind or in virtue of actual or possible human mental activity. On this view, we *make* some propositions true or false, in the sense that the structure of the human mind is somehow constitutive of mathematical truth. The view here is an idealism of truth-value, of sorts. It does not follow that we *decide* whether a given proposition is true or false, just as an idealist about physical objects holds that we do not decide what perceptions to have.[11]

An *idealism* in truth-value, then, seems midway between a realism and an anti-realism, insofar as one can distinguish between truth and mere fiction.[12] (When we bring this section on anti-realism to a close, we will take up the view which lies at an extreme—beyond *such* an anti-realism—namely, *fictionalism*.)

Along with Hilary Putnam, W. V. O. Quine has the reputation of defending a kind of numerical realism.[13] The *indispensability argument* is that one cannot do science without numbers and science is real, *ergo* numbers are real.[14] In response, Hartry Field published "Science Without Numbers," a defense of his own mathematical *fictionalism*.[15]

[11] Shapiro, *Thinking About Mathematics*, 29–30.

[12] One finds a parallel to this in Berkeley's idealism. Berkeley *seems* genuinely puzzled as to why the inability to distinguish reality from fiction would befall a man who holds that the *esse* of unthinking things is their *percipi*. See his defense of the distinction in *Treatise Concerning the Principles of Human Knowledge*, §§28–33 and *Three Dialogues Between Hylas and Philonous,* Dialogue Three.

[13] A platonism, in fact. See Arthur W. Collins, "On the Question 'Do Numbers Exist?'," *The Philosophical Quarterly* 48 (Jan. 1998): 23–36, especially 25–28. We will turn to Quine-Putnam shortly.

[14] *Real* here means only that one's ontological commitment to mathematical objects (like numbers) is *at least as strong* as one's ontological commitment to the physical entities they measure. Such a commitment, notes Collins, is not incompatible with declaring the whole of mathematics—even the whole of physical science—a myth.

[15] Paul Benacerraf supplied a *different* response in "What Numbers Could Not Be," a defense of his own *structuralist* account of mathematics. Some writers eventually raised the question whether perhaps even Aristotle is a mathematical fictionalist—though that is not a conclusion to which one would be naturally inclined. While fictionalism is decidedly anti-platonist, it suggests an exclusion of any sort of realism, Aristotle's included. Mathematical fictionalists often point to *Science Without Numbers* as a point-of-origin for the contemporary debate. Field argues that mathematics is an unnecessary extension to our physical theories, that is, useful, but not required. Mathematical objects are therefore *useful fictions*. For more on Aristotle as realist or fictionalist, see Julia Annas, *Aristotle's Metaphysics Books* M *and* N (Oxford: Clarendon Press, 1976), Penelope Maddy, *Realism in Mathematics* (Oxford: Oxford University Press 1990), and Edward Hussey, "Aristotle on Mathematical Objects," in "Peri tôn Mathêmatôn," ed. I. Meuller, *Apeiron* 24 (Dec. 1991): 105–34.

Field's counter to the *indispensability argument* is simple: science *can* be done without reference to numbers. For their part, Quine and Putnam see mathematics as *necessary* to the extent that scientific formulas *reference abstractions and quantify over them*.[16] Putnam supplies Newton's law of gravitation as a case-in-point.

> If the numericalization of physical magnitudes is to make sense, we must accept such notions as function and real number; and these are just the notions the nominalist rejects. Yet if nothing really answers to them, then what at all does the law of gravitation assert? For that law makes no sense at all *unless we can explain variables ranging over arbitrary distances (and also forces and masses, of course)*.[17]

As to whether science can do its work without invoking such abstractions, Field is willing to concede that it, in fact, *does* use mathematics in its work; still, it *need* not.

It is beyond the scope of this book to delve much further into Field's defense of his fictionalism. We can note that it entails reformulating at least one commonly held scientific formula in so-called *nominalistic* language and then showing that the nominalistic formulations follow from a physical theory, whether that theory is expressed mathematically or not.[18] It is easy to see how this account reduces to a 'useful fictions' or 'instrumentalist' claim.

As with *instrumentalism* in science, Field's anti-realism stands in sharp contrast to the claims laid down by Aristotle and Thomas Aquinas.[19] The very genesis of Aristotle's account of *episteme* lies in its connection to reality—one of the pre-conditions for *episteme* is that its subject must already be known to be real.

> It is necessary to have preexistent knowledge of two sorts: for in some cases it is necessary to assume that something is, in others what the term is, and in still others both. For example, of *concerning everything, either the affirmation or the denial is true* [we assume] that it is, of *triangle* [we assume] what it means, and of *unit* [we assume] both what it means and that it is.[20]

[16] Putnam's case-in-point is Newton's law of gravitation.

[17] Putnam, *Philosophy of Logic* (New York: Harper Torchbooks, 1971), 43. Emphasis mine.

[18] See Shapiro, *Thinking About Mathematics*, 229–37.

[19] Indeed, this applies to mathematical anti-realisms of whatever sort.

[20] *Analytica Posteriora*, I 1 71a11–16. διχῶς δ᾽ ἀναγκαῖον προγινῶσκειν· τὰ μὲν γάρ, ὅτι ἐστι, προϋπολαμβάνειν ἀναγκαῖον, τὰ δέ, τί τὸ λεγόμενόν ἐστι, ξυνιέναι δεῖ, τὰ δ᾽ ἄμφω, οἷον ὅτι μὲν ἅπαν ἢ φῆσαι ἢ ἀποφῆσαι ἀληθές, ὅτι ἐστι, τὸ δὲ τρίγωνον, ὅτι τοδὶ σημαίνει, τὴν δὲ μονάδα ἄμφω, καὶ τί σημαίνει καὶ ὅτι ἔστιν.

There is no *episteme* of mathematics without *such* a beginning in the real. Moreover, Aristotle incorporates *both* realisms: that of truth-functions (e.g., "*concerning everything, either the affirmation or the denial is true*," we know to be true) and that of ontology (e.g., "of *unit* [we assume] both what it means and *that it is*"). While intuitionism, as we saw, is similar to the realism of Aristotle and Thomas Aquinas in one key respect—each begins with an apprehension fundamental to the doing of mathematics—is does matter whether that apprehension is measured by a mind-independent reality, or rather happens independently as a mere consequence of being the kind of mind that it is. Try as one might, it is not possible to describe both intuitionist mathematics and Aristo-Thomistic mathematics as *knowledge* without equivocation. This first difference, then, between mathematical anti-realisms and the *episteme* of mathematics, could not be more fundamental.

Realisms

Conversely, of course, the view defended by Aristotle and Thomas Aquinas—being itself a realism—will have much more in common with contemporary forms of mathematical realism. We will briefly present two of those realisms here—Quine-Putnam indispensability and the structuralist account of Stewart Shapiro—with a view to showing (as above) in what respects the view we present and defend here is consistent with, or departs from, these other realisms. In the next chapter we will present professedly *Aristotelian and Thomistic* views of the philosophy of mathematics—views with which the present one will have more in common from the very outset.

Quine-Putnam indispensability is, as we said, a kind of platonism.[21] The term was introduced by Putnam in *Philosophy of Logic*.

> So far I have been developing an argument for realism along roughly the following lines: quantification over mathematical entities is indispensable for science, both formal and physical; therefore we should accept such quantification; but this commits us to accepting the existence of the mathematical entities in question. This type of argument stems, of course, from Quine, who has for years stressed both the indispensability of quantification over

[21] This is unqualifiedly so for Putnam. For his part, Quine distinguishes between *weak* and *strong* indispensability.

> mathematical entities and the intellectual dishonesty of denying the existence of what one daily presupposes.[22]

In his famous paper "Two Dogmas of Empiricism" and subsequent writings, Quine introduced his well-known "web of belief' analogy for scientific knowledge. Mathematics and logic lie near the center of this web though, famously, he grants that adjustments to the web—even complete inversions of its elements—are possible. Of course, the "web" contemporary and practicing scientists—as opposed to philosophers of science—have come to accept requires the use of mathematics, as we saw above in Putnam's account of Newton's law of gravitation—the *inverse-square* law. Thus the conclusion that one's *commitment* to abstract, mathematical objects must be at least as strong as one's commitment to distances, forces, masses, in short, to the physical objects of scientific study.

That the marker *abstract mathematical objects* enters the discussion (with good reason, of course—review Putnam's argument, above) is key. It implies, of course, a platonism in mathematics.[23]

It is not surprising that contemporary thinkers would look to the reason for the necessity and inalterability of mathematical claims elsewhere than in the physical universe—hence to a kind of platonism. While theories come and go, the mathematics indispensable to their expression is relatively stable. *Perhaps* this is because of its central place in science's web of belief. Still, there may be another explanation.

Side-by-side with the *episteme* of mathematics is Aristotle's account of the *episteme* of the natural world, given its general structure *via* the *Physics* and descending to particular areas of study *via* the *De Anima*, *Metereologica*, *De Caelo*, and other works more specific than these. What the *general episteme of natural things* brings to human knowledge is, not a changeable account of changeable things, but an *unchangeable* one. Developments in the history of the experimental sciences might lead one to think that the best we can do in coming to know such things is to improve upon and revise past accounts. One need not go to the extreme of a "web of belief" to come to the conclusion that natural science—the ever-changing story

[22] Putnam, *Philosophy of Logic*, 347. For more on Putnam, Quine, Maddy, and indispensability, see L. Decock, "Quine's Weak and Strong Indispensability Argument," *Journal for General Philosophy of Science / Zeitschrift Für Allgemeine Wissenschaftstheorie*, 33, no. 2: 231–50.

[23] Again, science is decidedly *a posteriori*, whereas logic and mathematics are, by contrast, necessary and *a priori*. In our view, that practicing scientists tend to mathematical realism is due to arguments like the indispensability argument.

we have written over several millennia—is *a contingent* account of *contingent* things.

Heraclitus has a special place among his contemporaries. It was his assertions *re* the changeability of the natural world that led Plato to—more or less—abandon his pursuit of natural philosophy and focus upon the *more stable* undertakings of ethics and metaphysics.[24] As Aquinas astutely points out, directly associating the object known and the character of our knowledge is an error.

> In this matter Plato strayed from the truth in that, having seen that all knowledge takes place by means of a kind of similitude, he came to believe that the form of the thing known is necessarily in the knower in the same way as it is in the thing. But the form of the thing known is in the intellect *universally*, *immaterially*, and *unchangeably*—as is clear from the very activity of the intellect, which understands things universally and with a certain sort of necessity. This is because the mode of the activity corresponds to the mode of the agent's form. So, he concluded that the things which we understand must exist in the same way, namely, immaterially and unchangeably.
>
> But this is not necessary. For even in sensible things we see that a form is in one sensible thing differently than it is in another. For example, a thing may be more or less intensely white, or in one thing we find whiteness conjoined with sweetness, whereas in another it is not. In like manner the sensible form exists in the external thing *differently* than it does in the senses, which receive the forms of sensible things without their matter—receiving the color of gold without receiving the gold, for example. So, too, the intellect, in its own way receives the species of material and changeable bodies in an immaterial and unchangeable way. For the received is in the receiver according to the mode of the receiver. We must conclude, then, that, through the intellect, the soul understands bodily things immaterially, universally, and with necessity.[25]

[24] There are several dialogues we can point to in support of this claim. Cf. the *Cratylus* and its discussion of the changeability of natural things, the *Timaeus*' division of forms into eternal and passing, and—perhaps most obviously—the analogies of the sun, divided line, and cave in the *Republic*.

[25] *Summa Theologiae*, Ia 84 1 *corpus.* "Videtur autem in hoc Plato deviasse a veritate, quia, cum aestimaret omnem cognitionem per modum alicuius similitudinis esse, credidit quod forma cogniti ex necessitate sit in cognoscente eo modo quo est in cognito. Consideravit autem quod forma rei intellectae est in intellectu universaliter et immaterialiter et immobiliter, quod ex ipsa operatione intellectus apparet, qui intelligit universaliter et per modum necessitatis cuiusdam; modus enim actionis est secundum modum formae agentis. Et ideo existimavit quod oporteret res intellectas hoc modo in seipsis subsistere, scilicet immaterialiter et immobiliter.

Aquinas is simply following Aristotle's response to the Heraclitean-Platonic argument. In fact, their fundamental changeability does not render natural things incapable of being known—even at the level of *episteme*.[26] Necessary and unchangeable knowledge *on our part* does not translate to a like unchangeability *on the part of what we know*. But this is precisely what mathematical platonists—anyone, really, who recognizes and respects the difference between the mathematical statements we use to define and describe natural things and the claims we make about those things themselves—hold. Mathematical objects need not be *separate from the physical order of things* to be known with that special—even unique—sort of conviction and necessity we find in mathematics. In final terms, then, the arguments leading to a platonic realm of abstract numbers, planes, and even sets, do not rightly conclude, since one can know even the changeable in an unchangeable way. Mathematical realisms need not be platonic, then. Indeed, the account presented and defended by Aristotle and Thomas Aquinas has a need neither to posit entities other than physical things nor to give an account of *access* to such entities.[27] On the one hand, Putnam and Quine are entirely correct: one ought to afford to what is mathematically indispensable to science at least as much reality as to the things to which one applies them. Still, it may be that there is a deeper issue here—and so a deeper solution. In fact, mathematicals and physicals may be more alike than different when it comes to *where* such things exist, *how* we come to know them, and what the *character* of such knowledge might be.

As to Shapiro's own mathematical views, he is very clear, calling himself a "thoroughgoing realist[s], holding both realism in ontology and realism in truth-value."[28] While structuralists are all *truth-value* realists, not all

Hoc autem necessarium non est. Quia etiam in ipsis sensibilibus videmus quod forma alio modo est in uno sensibilium quam in altero, puta cum in uno est albedo intensior, in alio remissior, et in uno est albedo cum dulcedine, in alio sine dulcedine. Et per hunc etiam modum forma sensibilis alio modo est in re quae est extra animam, et alio modo in sensu, qui suscipit formas sensibilium absque materia, sicut colorem auri sine auro. Et similiter intellectus species, corporum, quae sunt materiales et mobiles, recipit immaterialiter et immobiliter, secundum modum suum, nam receptum est in recipiente per modum recipientis. Dicendum est ergo quod anima per intellectum cognoscit corpora cognitione immateriali, universali et necessaria."

[26] Recall among Aristotle's conditions for *episteme* that what we know *cannot be otherwise*.

[27] The question of *access*, first raised by Aristotle regarding Plato's forms *in general*, remains one of the most vexing difficulties faced by contemporary mathematical platonists.

[28] Shapiro, *Thinking About Mathematics*, 32.

agree about the reality of mathematical *objects*.[29] As to *structuralism* itself, it is—as its name suggests—about abstract *patterns* obtaining among numbers. Shapiro (and presumably all the structuralists) deny that the numbers themselves have a reality *independent of one another*.[30]

At issue with Shapiro's structuralism—and with the Aristotelian realism of Franklin, to which we will turn in the next chapter—is whether one has simply substituted *patterns* for *numbers*, along with all the difficulties (for example, the problem of access), that accompany *arithmetical* realisms. On the other hand—and in *favor of* a mathematics of structure—there is something to be said for a study of real relations (beyond equality and inequality) which are necessarily consequent upon continuous quantity in three dimensions.[31] Even so, there is a difference between structure as a real relation consequent upon real quantities and structure as a real (i.e., independent) relation not consequent upon real quantities. The latter is Platonic, the former Aristotelian.

> Various positions in the extensive literature on universals delimit options for structuralism. One view, traced to Plato, is that at least some universals exist prior to and independent of any items that instantiate them. Even if there were no people and no red things, the properties of personhood and redness would still exist. This view is sometimes called *ante rem realism*, and universals so construed are *ante rem universals*. Ante rem universals (if such there be) exist prior to (and so independent of) the objects that have the universal. On this view, a 'one-over-many' is ontologically prior to the 'many'. … An alternative to ante rem realism, attributed to Aristotle, is that universals are ontologically dependent on their instances. On this view, there is no more to redness than what all red things have in common. Get rid of all red things and redness itself goes with them. Destroy all people and there is no longer such a thing as personhood. Universals so construed are called *in re universals*, and the Aristotelian view is sometimes called *in re realism*. Typical advocates of this view admit that universals exist, after a fashion, but they deny that universals have any existence independent of their instances. In a sense, the universals exist only in their instances. Ontologically, the 'many' comes first, and only then the 'one-over-many.'[32]

[29] Shapiro notes that other structuralists, Benacerraf and Hellman, "…do not presuppose the existence of mathematical objects." Ibid., 257.

[30] Ibid., 258. Also 261: "…it is nonsense to contemplate *numbers* independent of the structure of which they are part."

[31] We mentioned *handedness* from topology, earlier.

[32] Ibid., 262–263.

Shapiro is well aware that the issues and difficulties which arise in seeing pure mathematics as "the deductive study of structures as such"[33] parallel those arising in *any* realist mathematical account.

> There are two interrelated questions concerning the ontology of structuralism. One concerns the status of structures themselves. What is the natural number structure, the real number structure, and so on? Do structures exist as objects in their own right? What of more earthly structures and patterns, like a chess configuration, a basketball defence, or a symphony? The other group of issues concerns the status of individual mathematical objects, the places within the structures. What is the structuralist to say about numbers, geometric points, sets, and so on?[34]

We ourselves recognize the positive advance of extending mathematics to a consideration of structures—Shapiro places Euclidean geometry and topology side-by-side, as having specifically *spatial* structures in common. Still, what Shapiro sees as a strength we regard as weakness. True, defining mathematical objects *wholly* in terms of their *ante rem* structures sidesteps the question of platonic intermediates and their associated difficulties. Perhaps it is not a question of whether we study objects or structures of objects, rather, the distinction is finally a relative one. Whether we identify as the object a position within a structure (a bishop in a chess game, for example) or the structure itself (the game) is, says Shapiro, *more or less* arbitrary. Most importantly, *structures can exemplify themselves.*

> For the ante rem structuralist, then, the distinction between office and office-holder—and so the distinction between position and object—is a relative one, at least in mathematics. What is an object from one perspective is a place-in-a-structure from another. In the places-are-offices perspective, the background ontology can consist of places from other structures, when we say, for example, that the negative whole real numbers exemplify the natural number structure, or that a Euclidean line exemplifies the real number structure. Indeed, the background ontology for the places-are-offices perspective can even consist of the places of the very structure under discussion, when it is noted that the even natural numbers exemplify the natural num-

[33] Ibid., 259.

[34] Ibid., 261. Beyond these problems, Shapiro turns to the *access* question immediately following his consideration of ante- and in-rem structuralism.

> ber structure. In particular, each structure exemplifies itself. Its places, construed as objects, exemplify the structure.[35]

We mentioned above that the historical Plato is vulnerable to arguments of a certain form (i.e., the Third Man). Shapiro is well-aware of the objection—indeed, it is an objection Aristotle and Thomas Aquinas themselves would raise—and responds accordingly. After all, structures *do* exemplify themselves.

> Michael Hand argues that ante rem structuralism falters on a version of the traditional Aristotelian 'Third Man' argument against ante rem universals. Both the von Neumann and the Zermelo reductions exemplify the natural number structure. From the ante rem perspective, the natural number structure itself also exemplifies the natural number structure. Hand argues that the ante rem structuralist thus needs a *new* structure, a super natural number structure, which the original natural number structure shares with the von Neumann and Zermelo systems. And a regress emerges. From the ante rem perspective, however, the sentence 'the natural number structure itself exemplifies the natural number structure' turns on the different orientations toward structures. The idea is that the places of the natural number structure, considered from the places-are-objects perspective, can be organized into a system, and this system exemplifies the natural number structure (whose places are now viewed from the places-are-offices perspective). The natural number structure, as a system of places, exemplifies itself, as does every structure.[36]

Shapiro's insight—and that of the structuralists in general, it would seem—is that mathematics does not really study objects at all, that *structures* are a very different type of thing altogether—certainly different from standard *universals*—and so not prone to the type of arguments against Plato and platonists generally. It is not a question of wholes and parts, sets and members, types and tokens, but *structures themselves* (after all, each of the above *is* a structure). Of course, to study the *kind* of things structures are is to reenter a problematic realm (there would be *structure* as a universal and particular structures which instance that universal, and so on). But studying what type of thing a structure is, is not the same as studying *the*

[35] Ibid., 269. Shapiro distinguishes his view as being less relativistic than that of Michael Resnick—with whom he nevertheless has much in common. Resnick apparently sees the position-structure distinction along the lines of a Quinean web of belief. Cf. ibid., 267.

[36] Ibid., 269–70.

structure. We have seen where taking wholes as parts and sets as members can lead. Shapiro's claim is that recognizing the *sui generis* character of structures both avoids those problems and provides mathematicians with a coherent foundation for their work.[37]

In what respects, then, is the mathematical realism of Aristotle and Thomas Aquinas like, and unlike, Stewart's structuralism? In the first place, of course, it *is* a realism, one more straightforwardly *realist* than Quine-Putnam indispensability, as we have seen. There is something about the study of structure *via* Shapiro's examples of Euclidean geometry and topology that speak to and explain the unchangeability and necessity of spatial relations. As we noted above, however, there is a considerable difference between *structure* as a real relation *consequent upon* real quantities and structure as an ante rem relation. Recall that, for Aristotle and Thomas Aquinas, the very truth behind the claims of an *episteme* lies in its foundation in the real—in its acknowledgment of the *order* of the real. As even Shapiro grants, structures are *relational*. But relations are accidents *consequent upon* other accidents, whether quantity (and so arises the *episteme* of mathematics) or action and passion (leading to such relations as *cause and effect* in natural philosophy). It is one thing to abstract from—to "ignore," in Shapiro's own words—aspects of a relation or set of relations that are accidental to the relation as such. It is quite another to distance such relations from their real principles. It is not the case that—even given real space as a three-dimensional continuum and even given a platonic set of real numbers—there is an abstract structure exemplified in each, such that, were space and the set of real numbers not themselves *real*, the structure would remain. At issue is an *inversion* of the orders of thought and reality.[38]

What of an *Aristotelian*—in Shapiro's terms, an *in re*—structuralism?[39]

[37] For more on the uniqueness of structures, see Geoffrey Hellman, "Structures as *Sui Generis* Universals" in *The Oxford Handbook of the Philosophy of Mathematics and Logic*, ed., Stewart Shapiro (Oxford: Oxford University Press, 2005), chapter 17, §3, 541–6.

[38] Cf. Aquinas' observation, above, regarding the platonists' according to reality characteristics belonging solely to the intellect.

[39] Shapiro's consideration of *in rem* structuralism is quite brief and framed mostly in terms of set theory. Given the requirement of what he calls its *background ontology* (the *rem* of the *in rem* structures), his chief difficulty seems to be its *size*. "…an eliminative structuralist account of arithmetic requires an infinite ontology. Similarly, an eliminative structuralist account of real analysis and Euclidean geometry requires a background ontology whose cardinality is at least that of the continuum. An eliminative account of set theory requires even more objects. Otherwise, the fields are vacuous." Shapiro, *Thinking About Mathematics,* 272. Shapiro does consider—but ultimately rejects—a modal option.

One school of thought showing up neither in Stewart's *Thinking About Mathematics* nor in his *Oxford Handbook of Philosophy of Mathematics and Logic* is the so-called *Sydney School*, a group of Australian philosophers who propose and defend an explicitly Aristotelian and structuralist account of mathematics. Our guiding principle in presenting these recent mathematical views has been to make our way from those whose account is—*prima facie*, at least—most dissimilar to the *episteme* of mathematics to those with whom it is likely to be more in accord. In the next chapter, then, we will turn to contemporary Aristotelian and then Thomistic accounts of the nature of mathematics, asking once again whether and to what extent the position we have presented and defended here is like or unlike those.

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CHAPTER 13

Modern Aristotelian and Thomistic Accounts

The Aristotelians

Some philosophers of mathematics argue that Aristotle himself is an anti-realist.[1] Among those who support an Aristotelian philosophy of mathematics, Jonathan Lear is quite clear as to the opposite view.

> Let us use the word 'Platonist' to describe the position in the philosophy of mathematics held by Plato and his followers in the Academy. Let us use 'platonist' to describe anyone who believes that mathematical statements are true in virtue of the existence of abstract objects which exist outside space and time. (Kurt Gödel is an example of a platonist.) Finally, let us say that a 'mathematical realist' is someone who believes that mathematical statements are determinately true or false independently of our knowledge of them. Then one can say that Aristotle defends a form of mathematical realism while denying both Platonism and platonism.[2]

[1] We noted this above. We have not separately addressed this controversy, as our extensive consideration of Aristotle's mathematical realism in the first part of this book resolves that issue—we think—from the very start.

[2] Lear, "Aristotle's Philosophy of Mathematics," 188 note 35. It turns out that Lear ultimately means by *realism* something different from what Plato and Aristotle, and even Quine and Putnam, mean.

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J. W. Rioux, *Thomas Aquinas' Mathematical Realism*,
https://doi.org/10.1007/978-3-031-33128-2_13

Though Lear's Aristotle does not support the existence of *abstract* mathematical objects as the basis of the truths of mathematics, there *is* nevertheless—in direct response to Field and other mathematical anti-realists—such a basis. On the other hand, by Lear's account, the basis is not that its objects actually exist in physical reality,[3] rather, Aristotle tries to show how geometry and arithmetic make true in a way *which does not depend upon the singular terms having any reference or the quantifiers ranging over any separated mathematical objects.* The key to explaining the truth of a mathematical statement, says Lear, lies in explaining how it can be *useful*.[4]

There are two steps in Lear's argument. The first entails the seemingly insuperable objection to *any* sort of anti-platonic mathematical realism, the precision problem. The second is Lear's observation that the precision problem—and questions of physical mathematical objects—is not to-the-point anyway. Indeed, Lear says it is *unclear* whether Aristotle requires physical lines, circles, and spheres to have the properties which are proved of their geometrical counterparts.[5] Regardless, he is unconcerned, as the question of mathematical truth does not lie in a correspondence between these two realms anyway:

> There are two features of mathematical experience to which Aristotle's theory does not seem to do justice. First, much of mathematics, for example set theory, cannot easily be thought of as an abstraction from any aspect of physical experience. Second, there is the plausible belief that mathematical theorems are true irrespective of whether there is any physical instantiation of them.[6]

Lear is right. Aristotle's requirements for *episteme* do require that the subject about which we demonstrate necessary conclusions must first be known to exist. Moreover, he very clearly appeals to abstraction as being constitutive of the objects proper to mathematic, as evidence his seminal consideration of the difference between mathematics and natural philosophy in *Physica* II 2.

[3] Lear's 'truth-value realism' (to borrow a term from Shapiro) sees *real applicability* as a marker for which of our mathematical claims are true and which cannot (yet) be said to be true. Cf. Lear, "Aristotle's Philosophy of Mathematics," 191.

[4] Ibid., 188. Emphasis mine.

[5] In fact, the very opposite seems quite clear. Cf. Metaphysica, III 2 and XI 1.

[6] Lear, "Aristotle's Philosophy of Mathematics," 186.

Lear eventually concludes that it is in virtue of 'bridging' that mathematical claims are *true*:

> Thus, to explain the usefulness and applicability of mathematics we have had to follow Aristotle and appeal far more strongly to the existence of a bridge between the physical world and the world of mathematical objects than we have to the fact that mathematics is a conservative extension of science. The conservativeness of mathematics was invoked by Field to explain why we need not think of mathematics as true, but only as consistent. But to explain why this particular consistent theory rather than others is useful we have had to rely rather heavily on the existence of bridges and thus, I think, to *reimport* the notion of truth. For, in an Aristotelian spirit, one can allow that '2 + 2 = 4' is true without having to admit that there exist numbers in a Platonic realm outside of space and time. That there must exist bridges between the physical world and those portions of mathematics which are applicable to it implies that the mathematics must reproduce (to a certain degree of accuracy) certain structural features of the physical world. It is in virtue of this accurate structural representation of the physical world that applicable mathematics can fairly be said to be true.[7]

Aristotle's conception of *episteme* has been left far behind. For it begins with an apprehension of the nature of its ultimate subject, deducing universal and necessary properties of that subject in due fashion. Truth in such a system is not established *post hoc* but *a priori*. One's principles are true, one's subject is real, and one's reasoning is valid. There need *be* no subsequent check upon the results in a sound argument.

One direct consequence of Lear's *bridging* condition is that we can only deem a given mathematical claim true once we have demonstrated the 'marker' of its applicability. True, mathematics has as its basis in the structure of physical reality, but as we do not see perfect circles, lines, and spheres instantiated there, that their properties really express that structure cannot be assumed but must be *shown*, by way of the *bridge* of applicability. It follows, says Lear, that the line between *fictional* and *true* is ever-changing and—for all we know—cannot be *finally* drawn.

> Where is one to draw the line between the truth of mathematics and the parts of mathematics that are consistent with them, but neither true nor false? Nowhere: for there is no demarcation between applicable and nonapplicable

[7] Ibid., 190–1.

> mathematics that can be made with any certainty. One cannot determine a priori that a portion of mathematics is not applicable. It is conceivable, though unlikely, that we should discover that the world is sufficiently large and dense that we need a large cardinal axiom to describe it. But this does not mean that we should treat all of mathematics as true or all of it as merely consistent. That we cannot draw a distinction precisely does not mean that there is no distinction to be made: it means only that any suggested boundary will remain conjectural and subject to revision. Thus, the question of how much of mathematics is true can only be answered a posteriori.[8]

This is clearly *not* the mathematics Aristotle describes. *Episteme* need not await such a contingent judgment to pronounce mathematical conclusions to be true and known to be so. For his part, Lear acknowledges this.

> Though not Aristotle's, this philosophy of mathematics is Aristotelian. One of its virtues is that it does provide a harmonious account of truth and knowledge. Of those portions of mathematics that are not true, the question of knowledge does not arise. Of those portions that are true, there exist bridges of a fairly direct sort between the physical realm and the mathematical. And it is in virtue of our understanding of how these bridges link the mathematical and the physical that we can be said to know mathematical truths.[9]

The problem is clear. Both Lear's applicability criterion and Quine-Putnam indispensability propose to defend mathematical realism in the light of a perennial difficulty, but each does so at the expense of its *realism*. For our part, while we acknowledge the precision problem,[10] we deny that either applicability (or indispensability) is a condition both necessary and sufficient for mathematical truth—to say that a thing is real because it works or is needed for other things to work is certainly not to speak of the real in its plain sense.[11] As we have argued above, one can avoid fictionalism *and* maintain the conditions required for *episteme*. No small part of the solution lies in distinguishing among types of *the real*. Not only is it *not* the case that mathematical realism waits upon

[8] Ibid., 192.
[9] Ibid.
[10] We have proposed its solution in Chaps. 5 and 6, above.
[11] See Quine's own account of the matter in W.V.O. Quine, "On What There Is," 21–38.

applicability—mathematics becomes true only *a posteriori*—but, the conclusions of demonstrations are already *virtually contained* in their principles, as are subsequent *objects* of demonstration. As we argued above: mathematics is already *virtually complete*.

In April 2014 James Franklin published a brief *apologia* for Aristotelian realism in mathematics *via* the online source *Aeon*.[12] Clearly written for a popular audience, Franklin draws the rough outlines of a philosophy of mathematics which we find very familiar. Midway between Plato and nominalism, an Aristotelian account of mathematics, says Franklin, "…emphasises the way in which mathematical properties crop up in the actual world."[13] Briefly addressing both the epistemological and ontological benefits of such a realism, he even alludes to the notion of *episteme*, in that "[a] version of it, holding that mathematics is the 'science of quantity', was actually the leading philosophy of mathematics to the time of Newton, but the idea has been largely off the agenda since then."[14]

The points of comparison between this narrower *episteme* of quantity and Franklin's broader—albeit Aristotelian—view are many. There are real, that is extra-mental, accidents inhering in physical substances. Our capacity to re-present (imagine) such aspects of things in abstraction from their concrete conditions ushers us into the mathematical real. That we can then demonstrate certain properties of such subjects brings us to *episteme* proper, along with its accompanying necessity. Indeed, for Franklin, it is the *necessity* and *objectivity* of mathematical claims and theorems which argue most favorably for this kind of realism. Access is *via* sensation (much like the empirical sciences). Imagination—coupled with demonstration—yields results immune to the most common objections brought against necessity in the standard empirical realm.

On the other hand, Franklin extends Aristotle's *historical* realism to realms the Stagirite might have excluded. Besides those belonging to *quantity as such*, mathematical subjects, notes Franklin, include "…many other quantitative and structural properties that are not themselves physical but can be realised in the physical world (and any other worlds there might be): flows, order relations, continuity and discreteness, alternation,

[12] James Franklin, "The Mathematical World," 2014, Aeon, accessed December 29, 2022, https://aeon.co/essays/aristotle-was-right-about-mathematics-after-all.

[13] Ibid.

[14] Ibid.

linearity, feedback, network topology, and many others."[15] As he briefly delves into two of them—*ratio* and *symmetry*—he notes of the latter that "…[it], like other mathematical properties, can have causal powers, unlike abstracta as conceived by Platonists."[16] Of course, this is significant as a response to nominalists and platonists alike, in that the very *unreality*—in the first case—and *immateriality—in the second—of* mathematicals is tied to their having no causal relationships, either with any existing things or with certain (i.e., physical) existing things.

We noted above when addressing modern extensions to the *episteme* of continuous quantity in three dimensions—topology, specifically—that, should topological properties accrue to such a quantity, abstractly and as such, then this sort of extension would be a *prima facie* legitimate addition to the Aristotelian and Thomistic account as we have presented it. That Franklin approves of this extension in a presentation of this decidedly Aristotelian account allows us to investigate this question directly, now. On the other hand, Franklin does seem to grant that the *historical* Aristotle has been left behind.[17] Key to our own question here, then, is whether Aristotle's 'science of quantity' is *un-Aristotelian* in restricting the *episteme of mathematics* to quantity itself, abstractly and as such, and the relations of equality and inequality alone. Perhaps Franklin's *reasons* for the extension are true to Aristotle's principles and not, as with Lear's account, *Aristotelian, though not Aristotle's.*

Like Shapiro, Franklin advocates a kind of structuralism.[18]

> Aristotelian, or non-Platonist, realism holds that mathematics is a science of the real world, just as much as biology or sociology are. Where biology studies living things and sociology studies human social relations, mathematics studies the quantitative or structural aspects of things, such as ratios, or patterns, or complexity, or numerosity, or symmetry.[19]

[15] Ibid.

[16] Ibid.

[17] See the above quote regarding the *science of quantity.*

[18] On the other hand, Franklin self-describes his structuralism as Aristotelian—something Shapiro would never do.

[19] James Franklin, "Aristotelian Realism," in *Handbook of the Philosophy of Science*: *Philosophy of Mathematics*, ed., Andrew Irvine, gen. eds., Dov Gabbay, Paul Thagard, and John Woods (Amsterdam: Elsevier, 2007).

Recognizing that earlier efforts to define mathematics as *a science of structure* ran into problems of vagueness, Franklin proposes that its general principles are conceptual:

> A property is purely structural if it can be defined wholly in terms of the concepts same and different, and part and whole (along with purely logical concepts). … What has been asserted is that there are properties, such as symmetry, continuity, divisibility, increase, order, part and whole which are possessed by real things and are studied directly by mathematics, resulting in necessary propositions about them.[20]

Franklin's notion is decidedly Aristotelian. Since mathematics is to be the science (*episteme*) of structures, Aristotle's requirements would include an apprehension of the essence of that ultimate subject. Mathematics, then, is—not the study of the accident of quantity, as Aristotle would have it, but the study of *sameness* and *difference* and *part* and *whole*.

One important advantage to a more general subject (and definition) is *inclusion* as its natural result. We have seen many historical attempts to extend—never to contract—the consideration of the traditional mathematical sciences. Natural numbers to integers to rationals to reals to imaginaries to complexes to transfinites—each step in the process *broadens* the consideration of the mathematician and renders the definition of number more inclusive.[21]

> A typical mathematical truth is that there are six different pairs in four objects … *The objects may be of any kind, physical, mental or abstract*. The mathematical statement does not refer to any properties of the objects, but only to patterning of the parts in the complex of the four objects. If that seems to us less a solid truth about the real world than the causation of flu by viruses, that may be simply due to our blindness about relations, or tendency to regard them as somehow less real than things and properties. But relations (for example, relations of equality between parts of a structure) are as real as colours or causes.
>
> The statement that there are 6 different pairs in 4 objects appears to be necessary, and to be about the things in the world. It does not appear to be about any idealization or model of the world, or necessary only relative to

[20] Ibid., 113, 121.

[21] Logicism is an obvious case in point, moving from numbers as quantities to numbers as sets.

> axioms. Furthermore, by reflecting on the diagram we can not only learn the truth but understand why it must be so. [22]

The *six pairs in four objects* theorem does seem independent of whether the objects in question are points, people, relations, or even concepts. Much like the abstraction which makes the *episteme* of mathematics possible—abstracting continuous quantity in three dimensions and the mathematical unit from physical substances—Franklin argues that we need not consider which objects are paired or any of *their* properties, but only the *fourness* and the *pairings*, which are *relations*. Can this property be defined wholly in terms of the concepts same and different and part and whole, along with logical notions? Assuming so,

> There is nothing in this proof except what Aristotelian mathematical philosophy says there should be—no arbitrary axioms, no forms imposed by the mind, no constructions in Platonist set theory, no impredicative definitions, only the necessary relations of simple structural universals and our certain, proof-induced insight into them.[23]

On the other hand, there is *more* to an Aristotelian philosophy of mathematics than these. There is *good reason* why some Aristotelians reject the *mathematical* legitimacy of extensions to number theory, non-euclidean geometries, and (possibly) topology and graph theory on Aristotelian grounds. Nor is Franklin ignorant of the difference between his view and the historical Aristotle.

> The first theory, the one that dominated the field from Aristotle to Kant and that has been revived by recent authors such as Bigelow, is that mathematics is the 'science of quantity.' The second is that its subject matter is structure.
>
> The theory that mathematics is about quantity, and that quantity is divided into the discrete, studied by arithmetic, and the continuous, studied by geometry, plainly gives an initially reasonable picture of at least elementary mathematics, with its emphasis on counting and measuring and manipulating the resultant numbers.[24]

[22] Ibid., 101. Emphasis mine.
[23] Ibid., 138.
[24] Ibid., 108.

Why go beyond Aristotle's science of quantity to a science of structure? While Franklin is not certain what relation obtains between quantity and structural relations, he is clear that *symmetry* and other such real-world patterns require a mathematical consideration.

> The relation between the concepts of quantity and structure are unclear and have been little examined. The position that will be argued for here is that quantity and structure are different sorts of universals, both real. The sciences of them are approximately those called by the (philosophically somewhat unsatisfying) names of elementary mathematics and advanced mathematics. That is a more exciting conclusion than might appear. It means that the quantity theory will have to be incorporated into any acceptable philosophy of mathematics, something very far from being done by any of the current leading contenders. *It also means that modern (post eighteenth-century) mathematics has discovered a completely new subject matter, creating a science unimagined by the ancients.*[25]

At issue for us is, not whether there can be a scientific study of relations other than equality and inequality—as they arise among continuous spaces in three dimensions and pluralities of units—but whether such studies share a common name.[26] Why call Franklin's two sciences *mathematics* in the first place?[27]

[25] Ibid., 109. Emphasis mine.

[26] Not the mere name, of course. The real question is whether Franklin's science of structure is the same *episteme* as the one so carefully defined by Aristotle (and later by Aquinas). It is important also to learn whether a science of structure—as understood by structuralists themselves—is compatible with the very *notion* of *episteme* in the first place. Might it not be that the proper consideration of certain real relations (including purely structural ones) belongs to one or another existing science? After all, Aristotle himself addresses the nature of relations in (at least) two separate works—the *Categories* (logic) and the *Metaphysics*. We will develop this notion further in the concluding chapter.

[27] We remain undecided about relations that belong to quantity, abstractly and as such, beyond those of equality and inequality. On the one hand they are arguably quite real—in this we agree with Franklin. Does a consideration of them belong properly to mathematics? Even Euler was in doubt about this. Interestingly enough, Aristotle considers at least *some* of the spatial relations in *Physica III 5*, connecting them to the nonmathematical realm of sensible substances. On the other hand, he establishes such a connection on the basis of his account of proper—natural—places, and it is not at all clear that *symmetry*, *handedness, orientation*, and *rotational direction* are defined in terms of natural places in the universe. Some work is currently being done on this question by John Brungardt. Cf. J. G. Brungardt, "World Enough and Form: Why Cosmology Needs Hylomorphism," *Synthese* 198, Suppl 11, (2021): 2795–827.

To say nothing of the novelty of a new—anciently unimagined—science of structure, Franklin has much to say about the *original* sciences of quantity themselves. For even they must be altered so as to accommodate what we have learned about *quantity itself* since Aristotle's day.

In the first place, then, Franklin is full-willing to grant that some "quantities … seem to have nothing inherently to do with structure,"[28] supplying discrete quantities themselves and their numerical relations (ratios) as his examples. This suffices to show that the science of quantity is not simply a *part* of the newly discovered science of structure. Still, the ancient science of quantity must itself be expanded. Arithmetic must be altered so as to include ordinals and the ancient, irreducible divide between the continuous and the discrete must be abandoned. We now know there are discrete spaces (indeed, space itself may *be* discrete) and there is continuity among numbers. We have even achieved the reduction of real quantity to number in measurement theory.

> The theory of the ancients that the science of quantity comprises arithmetic plus geometry may be approximately correct, but needs some qualification. Arithmetic as the science of discrete quantity is adequate, though as the Benacerraf example shows, the study of a certain kind of order structure is reasonably regarded as part of arithmetic too. … But geometry as the science of continuous quantity has more serious problems. It was always hard to regard shape as straightforwardly 'quantity'—it contrasts with size, rather than resembling it—though geometry certainly studies it. From the other direction, there can be discrete geometries: the spaces in computer graphics are discrete or atomic, but obviously geometrical. Hume, though no mathematician, certainly trounced the mathematicians of his day in arguing that real space might be discrete. Further, there is an alternative body of knowledge with a better claim to being the science of continuous quantity in general, namely, the calculus. Study of continuity requires the notion of a limit, as defined and made use of in the differential calculus of Newton and Leibniz, and made more precise in the real analysis of Cauchy and Weierstrass. On yet another front, there is another body of knowledge which seems to concern itself with quantity as it exists in reality. It is measurement theory, the science of how to associate numbers with quantities. … In summary, the science of quantity is elementary mathematics, up to and including the calculus, plus measurement theory.[29]

[28] James Franklin, "Aristotelian Realism," 110.
[29] Ibid., 111–112.

These alterations to the ancient 'sciences of quantity' are needed if we are to accommodate these historical advances.

Finally, there are the new sorts of *quantitative relation*, called *structures*—the domain of *advanced mathematics*.

> That leaves the 'higher' mathematics as the science of structure. It includes on the one hand the subject traditionally called mathematical 'foundations', which deals with what structures can be made from the purely topic-neutral material of sets and categories, using logical concepts, as well as matters concerning axiomatization. On the other hand, most of modern pure mathematics deals with the richer structures classified by Bourbaki into algebraic, topological and order structures.[30]

It is clear that what Franklin means by *Aristotelian mathematical realism* is not Aristotle's historical account of mathematics, but an account that is neither *nominalistic* (i.e., quantities, ratios, and purely structural properties are real, and in things) nor platonistic (quantities and their pure properties are not abstract realities and universals are *in re* not *ante rem*).[31] To our way of thinking, however, though Franklin has made a good case for the reality of relations beyond those of equality and inequality, it is also clear that his mathematics does not remain an Aristotelian one—Aristotle had *much* more to say about the *third speculative episteme* than that its objects are real accidents in things.

There is one point of departure from the historical Aristotle that Franklin sees as more fundamental and decidedly in his favor, which is the necessity of mathematics. Taking this as a sort of *distinctive difference* between mathematical and other forms of reasoning, Franklin takes Aristotle to task for a failure in addressing the question that matters most: whence its necessity?

> An essential theme of the Aristotelian viewpoint is that the truths of mathematics, being about universals and their relations, should be both necessary and about reality. Aristotelianism thus stands opposed to Einstein's classic dictum, 'As far as the propositions of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.'…

[30] Ibid., 112.

[31] Franklin contrasts his own structuralism to Shapiro's own on precisely this point. See Ibid., 108.

> Mathematics provides, however, many prima facie cases of necessities that are directly about reality. One is the classic case of Euler's bridges, mentioned in the previous section. Euler proved that it was impossible for the citizens of Königsberg to walk exactly once over (not an abstract model of the bridges but) the actual bridges of the city.[32]

For Franklin, it is the absolute necessity of its claims that makes mathematics to *be* mathematics. In this he is in accord with Aristotle, who says in many places that the certitude of mathematical claims distinguishes them, even from the claims made in the other forms of *episteme*. Franklin's account has a superiority over the limited account Aristotle provides on an *epistemological* basis.

> What exactly is the relation between the mind and universals, the relation expressed in the crude metaphor of the mind 'grasping' universals and their connection? 'Insight' (or 'eureka moment') expresses the psychology of that 'grasp', but what is the philosophy behind it? Without an answer to that question, the story is far from complete. It is, of course, in principle a difficult question in epistemology in general, but since mathematics has always been regarded as the home territory of certain insight, it is natural to tackle the problem first in the epistemology of mathematics.

Absent a cogent account of the *necessity* of mathematical claims, Aristotle is in an awkward position. After all—as Franklin points out—being a realist, Aristotle sees no fundamental difference between the *episteme* of natural philosophy and that of mathematics. Indeed, as we ourselves saw, he needs to *argue for* a distinction between the two, since they are about the same real things.

> Aristotle says that mathematicians differ from physicists (in the broad sense of those who study nature) not in terms of subject-matter, but in terms of emphasis. Both study the properties of natural bodies, but concentrate on different aspects of these properties. The mathematician studies the properties of natural bodies, which include their surfaces and volumes, lines, and points. The mathematician is not interested in the properties of natural bodies considered as the properties of natural bodies, which is the concern of the physicist. Instead, the mathematician is interested in the properties of natural bodies that are 'separable in thought from the world of change'.

[32] Ibid., 113.

> But, Aristotle says, the procedure of separating these properties in thought from the world of change does not make any difference or result in any falsehood.[33]

This is as fair an account of Aristotle's own distinction between mathematics and natural philosophy as one might wish. Still, says Franklin, why should a consideration of these aspects of natural things rather than those result in a *higher degree of certitude* in the one case? It is Aristotle's *epistemology* of the real that Franklin calls into question. Franklin is quite doubtful of the cogency of what has come to be called *intentional reception* as an explanation for knowing things as they are.

> …Based on Aristotle's dictum that 'the soul is in a way all things', the scholastics maintained that the relation between the knowing mind and the universal it knows is the simplest possible: identity. The soul, they said, knows heat by actually being hot ('formally', of course, not 'materially', which would overheat the brain).
>
> That theory, possibly the most astounding of the many remarkable theses of the scholastics, can hardly be called plausible or even comprehensible.[34]

On the assumption that the knower is identical with the known, Aristotle and the scholastics argue, objects *abstracted from the changeable* would be more certain, *prima facie*, than those whose abstraction includes changeability.[35] For his part, Franklin sees this *identity* account as being implausible, to say the least.

Still, if mathematics is a study of quantities and structures, the *identity* account becomes more plausible and supplies an (at least workable) explanation for the very thing that distinguishes mathematics from other studies: its absolute necessity.

> That [identity] theory, possibly the most astounding of the many remarkable theses of the scholastics, can hardly be called plausible or even comprehensible. What could 'being hot formally' mean? Nevertheless, it [the theory] has much more force for the structural universals of mathematics than for physical universals like heat and mass. The reason is that structure is 'topic-neutral' and so, whatever the mind is, *it could in principle be shared*

[33] Ibid., 103–104.
[34] Ibid., 138–139.
[35] Cf. Aquinas' account of the platonists and changeability, above.

> *between mental entities (however they are conceived) and physical ones*. While there seem insuperable obstacles to the thought-of-heat being hot, there is no such problem with the thought-of-4 being four-parted (though one will still ask what makes it the single thought-of-4 instead of four thoughts).[36]

It is here that Franklin's account goes beyond being *merely realist* and ventures once again into the Aristotelian realm. For the principle that epistemological necessity follows upon necessity in the known if the identity theory holds is, after all, Aristotle's own. Taking Franklin's epistemological route: what constitutes mathematics is the absolute necessity of its claims, and knowledge which is necessary is the result of an identity *with* what is in itself necessary. But such an identity is only plausible for things which can be instantiated in the physical and mental world alike—quantities and structures—and not for ordinary physical characteristics, like heat and mass. Ergo the study of quantities and structures—being necessary in this way and known to be so—is a mathematical one.

Returning to Franklin's example of the *six pairs in four objects theorem*, his point is clear. "A typical mathematical truth is that there are six different pairs in four objects… *The objects may be of any kind, physical, mental or abstract.*"[37] The theorem itself can be mentally duplicated. If identity is *the* principle of knowledge—certain knowledge—then it is no wonder that we insist upon the absolute necessity of the theorem. We see it, as it were, with our mind's eye.

We have already stated—on Aristotelian grounds—our objections to some of the extensions Franklin accepts. Still, there is one such extension we have held in abeyance to this point—the real structural relations studied in topology. Franklin introduces the historical novelty of topology *via* the same example we used earlier, the bridges of Königsberg. As we left the question in Chap. 10, it comes down to whether topological relations are *quantitative* space-relations. Do *orientation*, *handedness*, and *rotational direction* follow upon continuous quantity in three dimensions, abstractly and as such?

Let us scrutinize Franklin's new *four objects must be in six pairs* theorem. Prescinding from the issue whether the objects in question are really *topic neutral*, let us assume—like Franklin—four points in Euclidean space

[36] Ibid., 139.
[37] Ibid., 101.

as our instance.[38] Beginning with the claim that there are *four* objects, we can find an analogy (at least) to a number of *parts* of space as it is treated in Euclidean geometry.[39] After all, for Euclid, all geometrical figures are bounded (solids by surfaces, plane figures by lines, and lines by points). A *triangle* is a plane figure bounded by three straight lines. A *square* is a plane figure bounded by four equal straight lines set at right angles.[40]

Taking the square as our example, then, it is defined by boundaries of two sorts, points and straight lines. In both cases there are four. Franklin's theorem is therefore instanced in a Euclidean square in two ways: *the four points bounding its sides must be in six pairs* and *the four sides bounding the square itself must be in six pairs*. We might have just as easily chosen the triangle as our figure to draw the conclusion that *the points bounding its straight sides must be in three pairs* and that *the three sides bounding the triangle itself must be in three pairs*. So, too, with a five-sided rectilinear figure—the five points bounding its straight sides must be in ten pairs. So too for its five sides.

It turns out that Franklin's *four objects must be in six pairs* theorem is a *particular* case of a general *pairing* theorem, which is itself a *particular* case of a general *grouping* theorem. Indeed, one need not look even as far as pairs and other groupings to discover the necessity of theorems not included among Euclid's own, for example, *an n-gon has n sides and n vertices*. Importantly, we think, this last is just another way of saying that *n-gons* are necessarily grouped into *n* groups of singletons—irrespective of whether we count vertices, edges, or angles. Given Euclidean space and *any four objects within it*, the notion that there will be only *four* singleton groups is *already known* by way of the definition of the polygon in question. This is to say that something obvious can be deduced from the definitions (those of the polygons) given—*whether Euclid saw fit to deduce it or not*.

It is unclear whether *our* theorem—*an n-gon has n sides and n vertices*—is just the sort of thing Franklin has in mind. Clearly, it would take more thought to generalize from the *singleton* theorem to theorems about

[38] A *red flag* that immediately arises is that the *discipline* of topology does not restrict itself to Euclidean spaces—it is very much unclear what topological theorems would find their 'necessity in truths directly about the real world' outside the topology of Euclidean space. We are assuming that Franklin would restrict the science of structure—to the extent that it is *really* instanced in space—to Euclidean spaces and not all topological spaces.

[39] Taking the word *parts* very broadly.

[40] Cf. Euclid, *The Elements*, I Definition 22 and I 46.

pairings to theorems about *n*-groups in general. Our point here is only that our theorem seems *already to be contained* in Euclid's geometrical principles. Perhaps topology is not so much a new science *unimagined by the ancients* as new *conclusions* that follow from the same principles as before. After all, the *necessity* of the singleton theorem is rooted in the very same reality of three-dimensional space which Franklin allows.

Of course, Franklin might not say he is constructing a quadrilateral in his *four objects have six pairings* theorem. After all, he insists that the *structural* theorem is independent of its instance—in his words, it is "topic-neutral." While he instances the theorem in four points on a Euclidean plane, he might very well have made the same point—he would say—with the four attributes of knowledge considered by Immanuel Kant in the Introduction to his *Critique of Pure Reason*: *a priori*, *a posteriori*, *synthetic*, and *analytic*. There are six and only six ways these attributes can be paired. So, too, with the group *a concept*, *right-handedness*, *bauxite*, and π. *Any* group of four—including an arbitrary group of four—admits of six and only six pairings. On the other hand, when we try to instance our *singleton* theorem by this example, we get a strange result. It is entirely *un*clear what type of *four* we are counting here. Four *whats*, exactly? Objects, presumably. Which is where—in our judgment—Franklin's account of the science of structures begins to break down. While the term *one* (or *object*, or even *structure*, for that matter) may seem in this context to be univocal—after all, the *four objects have six pairings* theorem begins with four objects, which presupposes each object is *one—there does not seem to be a unit among the members of the group we just proposed*. One *could* argue that Kant's foursome have a *kind* of unity—perhaps they are in the same genus as being qualities of thought—it is not clear that they are *four* in the *mathematical sense*: four *units*. As to our intentionally arbitrary group, one cannot discern *any* sort of unity among the members—as we said, four *whats*, exactly?

It is an Aristotelian commonplace to require univocal language in demonstrations.[41] It may be that the *terms* employed in Franklin's mathematics of structure are not *really* instanced in various ways. They *are* directly instanced in continuous quantity in three dimensions and pluralities of the mathematical unit, but it seems only by way of extension of the terms that they arise elsewhere.

[41] Else one encounters the *fallacy of equivocation*.

We have already seen a sense of the word *handedness* that would be excluded from consideration by Aristotle and Aquinas.[42] The notions whereby Franklin distinguishes purely structural properties as such from *concretely* structural ones—like a Tartan pattern or a basketball game—are very broad, it is true, and seemingly obtain *transcategorically*. On the other hand, *same different*, *part*, and *whole* are notoriously ambiguous. Indeed, Aristotle conceives of the *same* as a kind of unity *within* a category, namely, quality. *Unity of substance*, on the other hand, he calls *identity*, while—as we would expect, of course—unity among quantities is *equality*, while quantitative differences are *inequalities*. It is not clear in what respect four points in space as a stand-in for *any four objects at all* instances the *same* notions of *same*, *different*, *part*, and *whole*, along with logical notions.

In sum, at issue is whether Aristotle *himself* failed to recognize the greater inclusivity of the *episteme* of mathematics. Had he seen the commonality shared by the quantitative relations of equal and unequal with, say, topological relations proper to continuous quantity in three dimensions, even to the *sameness*-and-*difference* and *whole*-and-*part* relations shared by everything and anything, would he have seen mathematics as the *episteme* of *sameness*-and-*difference* and *whole*-and-*part*, *among which* are found quantitative relations of equality and inequality? Did Aristotle mistake trees for the forest?

First, we note that Franklin's own principle whereby modern extensions to the ancient 'science of quantity' are legitimated is *epistemological necessity*—the unsophisticated recognition, for example, that *four objects have six pairings*. And, while Aristotle himself takes such necessity as a point-of-difference between mathematics and other studies, he does not affirm the consequent—*if mathematics then necessary* does not equate to *if necessary then mathematics*. After all, the first principles of each form of *episteme* admit of the same sort of necessary—that is, they are known to be so—as do even the theorems of mathematics. What makes mathematics distinctive—and so admitting of the kind of necessity we all know it to possess—is the way it abstracts its objects. The royal pathway to mathematics lies along an *abstraction from matter and motion*. To the extent that topological properties do so, then, one might admit them into the

[42] That is, if *right-handedness* and *left-handedness* are intended, these are merely *mental* when seen as relations belonging to space, in abstraction and as such.

episteme of mathematics. On the other hand, we have seen that they may, in fact, be deducible from the principles which are already in place.

Second, we think that, finally, it is not true to say that quantities and purely structural properties are—in Franklin's words—"topic-neutral," that the structural universals of mathematics can be *literally* shared between physical objects and the mind. And key to this observation is the Aristotelian account of *equivocation*.

The Thomists

We turn, at last, to some of the modern Thomistic accounts of the nature of mathematics with a view, once again, to finding what we may have in common and where we may differ.

Historically, mathematical developments have run parallel to developments in other areas of human knowledge. Relatively *conservative* views give rise to thinkers who successfully challenge the rigid notions of their day. This trend is instanced in number theory, beginning with its narrowest scope in the natural numbers, followed by the hard-fought inclusions of zero, the integers, the rationals, the transcendentals and reals, and infinite and complex numbers.[43] Arguing for the *legitimacy* of such developments has become an integral part of mathematics itself, which we saw clearly when we considered the discovery of transfinites by Georg Cantor. It would be helpful to briefly review Cantor here, most especially as regards his view of free mathematical inquiry.

As we saw, Cantor holds that mathematics consists in developing methods for the construction of its objects and discovering the mathematical rules governing their relationships. His notion of the *reality* of mathematical objects[44] is twofold: they have both an *intrasubjective* (immanent) reality and a *transsubjective* (transient) reality. As to the *construction* of mathematicals, Cantor emphasized the *lack of restrictions* under which the mathematician ought to be placed as of paramount importance.

> On account of this singular position which differentiates it from all of the other sciences and affords an explanation for the comparatively easy and

[43] There is some evidence for a broader scope even in ancient times. Still, this paper is about attempts by mathematicians and philosophers to justify a freer range of legitimate mathematical objects, hence *hard-fought*.

[44] Indeed, he applies the distinction to the reality of all conceptualizations.

> unrestrained manner with which one may operate with it, it particularly deserves the name of *free mathematics* (*freien Mathematik*), a designation which, if I had my way, would take precedence over the usual expression 'pure' mathematics. … There is no need, as I believe, of fearing any danger to the sciences in these principles, as many think: the conditions imposed under which the freedom of constructing numbers can be exercised are of such a kind that they permit very little room for arbitrary action (*Spielraum*); for every mathematical concept carries within itself the necessary corrective; if it is fruitless and purposeless, that is soon seen and it will because of little success be abandoned.[45]

Restrictions placed upon free mathematical inquiries, then, are unwarranted. Moreover, such restrictions are commonly found in the philosophy of Aristotle and in scholastic developments of that thought.[46] Some agree with Cantor, while others argue that at least Thomas Aquinas—and possibly Aristotle—might very well have embraced such developments.[47]

Are these *scholastic* restrictions Cantor fears *necessarily* consequent upon the principles of traditional philosophy, or is it possible to legitimize more recent developments in number theory in light of those very principles? What is it *in* Aristotle and Aquinas which lead mathematicians like Cantor to reject them? As we saw, both argue that mathematics is *episteme*, and we arrive at mathematicals *via* a special form of *abstraction*. Once again, from Aristotle:

> The mathematician makes a study of things which result from abstraction, for he studies things by taking [from them] everything sensible, such as heaviness and lightness… and he leaves only quantity and the continuous

[45] Georg Cantor, *On Infinite, Linear Point-Manifolds*, 115.

[46] We have seen this specifically in regard to his development of the transfinite numbers, but the same point can be made for other such developments, for example, the real or imaginary numbers.

[47] As examples of the former, see Small, "Cantor and the Scholastics," 407–28 and Robert J. Spitzer, *New Proofs for the Existence of God*, (Grand Rapids, MI: Eerdmans, 2010), especially chapter 5, 177–80. Spitzer sees the rejection of actual infinities as the basis for a modified *kalam* argument for the existence of God—one which, he contends, Aquinas himself would have embraced. As examples of the latter, see Anderson, "Aristotle and Aquinas on the Freedom of the Mathematician," 231–55, Maurer, "Thomists and Thomas Aquinas on the Foundation of Mathematics," 43–61, and Drozdek, "Number and Infinity: Thomas and Cantor," 35–46.

> (according to one, two, or three dimensions) and the attributes of these as quantity and as continuous.[48]

To be honest, constraints upon the free inquiry of the mathematician—constraints Cantor fears—follow quite readily. If traditional philosophy is right, mathematicians need *justify* imagined—even if consistent—extensions to a simple, foundational theory are, in addition, real attributes of the physical world. Establishing such a correlation between legitimate mathematics and external reality may be more or less feasible when we consider early developments (e.g., the rationals), but it becomes increasingly (and obviously) less feasible as we move to the transcendentals, reals numbers, and (enter Cantor) transfinites.[49]

Some philosophers—even Thomists—argue that Cantor's *scholastic fears* (the fears of anyone desiring to avoid philosophical constraints to free mathematical inquiries) are unfounded and claim that traditional philosophy is very much open to the historical developments in mathematics of which its founders would have been unaware, but which they would have embraced *had* they been aware.

In making this case for the mathematical philosophy of Thomas Aquinas and perhaps even Aristotle, Thomas Anderson begins by raising the question.

> If mathematical quantities are nothing more than abstracted real quantities; if they are gained simply by 'stripping away' all nonquantitative attributes of things, does this mean that for Aquinas and Aristotle the mathematician is limited in his science to treating objects which in their quantitative features resemble the quantitative attributes of physical things? It is true that both men give as examples of geometrical objects rather elementary figures, circles, triangles, angles, etc., which could easily be gained by abstraction from similarly figured sensible things. But does this mean that they believe that

[48] *Metaphysica*, XI 3 1061a29–35. ὁ μαθηματικὸς περὶ τὰ ἐξ ἀφαιρέσεως τὴν θεωρίαν ποιεῖται(περιελὼν γὰρ πάντα τὰ αἰσθητὰ θεωρεῖ, οἷον βάρος καὶ κουφότητα … μόνον δὲ καταλείπει τὸ ποσὸν καὶ συνεχές, τῶν μὲν ἐφ' ἓν τῶν δ ἐπὶ δύο τῶν δ ἐπὶ τρία, καὶ τὰ πάθη τὰ τούτων ᾗ ποσά ἐστι καὶ συνεχῆ…) See also *Physica*, II 2 193b31–35. Aquinas provides an extended treatment of mathematical abstraction in his *In De Trinitate*, V 3.

[49] As we have seen, Cantor insists that the transfinite numbers exist outside our mind (as ideas in the mind of God). Still, establishing this connection is not presupposed to discovering the truths of mathematics. On the other hand, such constraints do not arise via ancient and scholastic philosophers alone. *The big three* mathematical philosophies of the late nineteenth and twentieth centuries arose as a result of this very issue.

> mathematics is limited to just such quantities, quantities which bear almost a one-to-one relation to real quantities? If this is the case, then clearly *the freedom of the mathematician is severely restricted*.[50]

Anderson concludes that though *Aristotle* might have had difficulty accepting such developments, Aquinas may not have—and even here, much depends upon the role Aristotle would have given to the *imagination*. While Anderson does find the imagination mentioned in Aristotle's writings, it is entirely unclear what place it has in mathematical reasoning, for him. Aquinas *does* see the imagination as the basis for resolutions to mathematical judgments:

> Our judgments about some things do not depend upon what we perceive by sense because, though they really do exist in sensible matter, they are abstracted from it according to their defining *ratio*, and we make our best judgment of a thing in accordance with its defining *ratio*. As to their defining *ratio*, since they do not abstract from just *any* matter but only sensible matter, and with matter removed from sensible conditions there remains something imaginable, therefore in such things we must judge according to what the imagination reveals. Mathematics is about such things. Therefore, in mathematics, what we know by way of our mathematical judgments must terminate in the imagination and not the senses, because a mathematical judgment goes beyond the apprehension of sense. So it is that sometimes a judgment we make about a mathematical line is not the same as one we make about a sensible line, for example, that a straight line touches a sphere at only one point, which is true of a separate [abstract] straight line but not of a straight line in matter, as is said in the first book of *De Anima*.[51]

[50] Anderson, "Aristotle and Aquinas on the Freedom of the Mathematician," 233–4. Emphasis mine.

[51] *In De Trinitate*, VI 2.corpus. "Quaedam vero sunt, quorum iudicium non dependet ex his quae sensu percipiuntur, quia quamvis secundum esse sint in materia sensibili, tamen secundum rationem diffinitivam sunt a materia sensibili abstracta. Iudicium autem de unaquaque re potissime fit secundum eius diffinitivam rationem. Sed quia secundum rationem diffinitivam non abstrahunt a qualibet materia, sed solum a sensibili et remotis sensibilibus condicionibus remanet aliquid imaginabile, ideo in talibus oportet quod iudicium sumatur secundum id quod imaginatio demonstrat. Huiusmodi autem sunt mathematica. Et ideo in mathematicis oportet cognitionem secundum iudicium terminari ad imaginationem, non ad sensum, quia iudicium mathematicum superat apprehensionem sensus. Unde non est idem iudicium quandoque de linea mathematica quod est de linea sensibili, sicut in hoc quod recta linea tangit sphaeram solum secundum punctum, quod convenit rectae lineae separatae, non autem rectae lineae in materia, ut dicitur in I De anima."

At root, this account of mathematical abstraction seems not to differ from what we have already said about Aristotelian *separation*. Indeed, the difficulties brought to Aristotle's philosophy of mathematics by *idealization* and *the precision-problem* find their solution *in* constructive demonstration—an explicitly Aristotelian tenet, as we saw above. Aquinas' reference to I *De anima* suggests that he, at least, discovers the same view in Aristotle's writings. Anderson leaves the question open:

> Since individuals are attained directly only by sense and not by mind which is directly of the universal, individual mathematicals could be grasped directly only by a sense faculty. But since the quantities studied in mathematics are not possessed of any proper sensible features, they cannot be grasped by the exterior senses. Would it, then, be imagination which grasps them? To be sure, Aristotle does speak in the *Metaphysics* of individual mathematicals as known by 'intuition.' … But is this intuition imagination? Some have so interpreted it; Aristotle himself does not say.[52]

But why make the point of departure the imagination? Anderson sees in the imagination a direct link to freedom of mathematical inquiry—the sort of freedom Cantor deemed so important to the mathematical process.

> …if individual mathematical objects have their locus in imagination, it would follow that there is a certain degree of freedom on the part of the mathematician in regard to his objects. The Stagirite himself refers in various places to the freedom men have in imagining. If the locus of individual mathematicals were the imagination it would seem to follow that the mathematician would be free to deal with objects which do not closely correspond to anything found in the physical world. There would be no reason to limit him to simply studying abstracted quantities which resemble the quantities of things, but he could treat quantities which he himself had devised in imagination which have no one-to-one correspondence to any physical quantities. Indeed, an epistemological basis could be provided for

[52] Anderson, "Aristotle and Aquinas on the Freedom of the Mathematician," 236. Anderson explicitly names Aquinas as being among those who have ascribed such a view to Aristotle. As to Aquinas himself on the importance of imagination in mathematics, see *In De Trinitate*, VI 2.

> the tremendous development in modern times of nonrepresentational mathematical systems such as the nonEuclidean geometries.[53]

The reasoning seems to be: the imagination has a certain amount of freedom in constructive demonstrations, whereas the abstraction of mathematicals along the lines required by Aristotle and Aquinas requires a conformity of the mind—or imagination—to the real object. On the other hand, if the (constructive) imagination is the foundation of mathematical judgments, then, mathematical judgments need not be limited to those mathematicals we can directly abstract from sensible substances. Mathematicians are thereby free to found legitimate conclusions on what the imagination reveals, which ranges far beyond what we directly experience.

Anderson continues:

> Our question is then is the mathematician free to use these basic elements to construct (and hence demonstrate the existence of) any figure he desires—any figure that is, whose very existence is not self-contradictory (like square circles)? Certainly the most basic abstracted elements, those whose existence is simply posited, are so general as to be able to form any figure or number. And yet the Stagirite never states that the mathematician has the freedom to construct these elements into any non-self-contradictory objects he pleases. In fact, it is just the opposite as we have said, the only objects of geometry he cites are those which closely resemble physical magnitudes. Could this indicate that he never thought of allowing the mathematician freedom to construct and treat objects not resembling quantified physical things? On the other hand, it might be suggested that Aristotle would never have intended such a limitation of mathematics since numbers by their very nature as more abstract than magnitudes are clearly not able to be closely bound to physical quantities.[54]

In the end, Anderson believes that Aquinas developed Aristotle's account of the mathematical sciences significantly. One such advance would be founding mathematical judgments on acts of the imagination, an indication of Aquinas' openness to free, even creative, inquiries on the

[53] Anderson, "Aristotle and Aquinas on the Freedom of the Mathematician," 237.
[54] Ibid., p. 238.

part of mathematicians.[55] Another such development is referenced in the *Scriptum Super Sententiarum* passage we considered above.[56] Following Maurer, Anderson, too, takes this singular text to indicate a possible development in Aquinas' account of the mathematical sciences—a departure from the rigid mathematical philosophy of Aristotle.

> I would like to suggest a contrast, or at least a difference in emphasis, between Aquinas and Aristotle on this point. The difference as I see it is that, compared to St. Thomas, Aristotle tends to view the mathematician as more passive in his act of abstraction. To be sure, he 'strips away' all the non-quantitative features of physical things—and this 'stripping' itself is an activity on his part. Yet when it comes to the actual grasping of physical quantity, the connotation is that the mathematician simply grasps what remains after all nonquantitative features are removed. He simply 'liberates,' so to speak, the real quantities of things from their sensible, mobile, material existence, and proceeds to study them—real quantities but not qua real. Now it is certainly true that Aquinas in many places, especially his Commentaries, speaks of the mathematician's abstraction in the same terms that his Greek predecessor uses. (See, for example, texts cited in my first Section.) Nevertheless, in the passage discussed in the previous paragraph he shows, I believe, more recognition of the activity of the intellect in the actual production of mathematicals. The mathematician does not just grasp real quantity stripped clean, he does not simply study a likeness of real quantities, rather his object is directly a product of his intellect's own activity—granted that the activity has its remote foundation in the experience of physical quantities.[57]

Maurer and Anderson, then, are of like mind:

> There are natural lines and circles but these are not the lines and circles of mathematics. There are also numbered or numerable multitudes in reality but they are not the numbers of mathematics. The objects of mathematics, though remotely based on the sensible world and abstracted from it, are products of a mental activity which alters their definitions and natures.

[55] Thus Anderson, regarding Aquinas: "As for the freedom of the mathematician in his demonstrations of existence, it would seem that he is at liberty to construct in imagination *any mathematicals he can*, and this would apparently mean any quantities whose existence is not self-contradictory." Ibid., p. 252. Emphasis mine.

[56] Again, cf. Maurer, "A Neglected Thomistic Text on the Foundation of Mathematics" and Maurer, "Thomists and Thomas Aquinas on the Foundation of Mathematics".

[57] Anderson, "Aristotle and Aquinas on the Freedom of the Mathematician," 243–4.

> Nothing like them exists in the external world. Aquinas was clearly aware of this from his knowledge of contemporary mathematics. Had he known modern types of mathematics, he would have seen *the almost unlimited range of the mind's mathematical inventiveness*.[58]

As is often the case, we should ask what is *at stake* in this controversy. If Aquinas does go beyond Aristotle in grounding mathematical judgments in the imagination, why *not* pronounce as legitimate whatever is imaginable? If the mathematicals are like logical intentions, *entia rationis*, why place any restrictions upon mathematical discoveries other than that such constructs be self-consistent?

Moreover, the *direct importance* of the imagination to mathematics is evident in Aquinas.

> Now there are certain things, our judgment of which does not depend upon what we perceive by sense, because, even though they exist in sensible matter in reality, nevertheless they are abstracted from sensible matter according to their defining *ratio*: but we make our best judgment of anything in accordance with its defining *ratio*. Now since they do not abstract from just any matter according to their defining *ratio*, but only from sensible matter, and with matter removed from sensible conditions there remains something imaginable, therefore in such things we must judge according to what the imagination reveals, and mathematics is about things of this sort. Therefore in mathematics the knowledge we have according to judgment must terminate in the imagination, not the senses, because a mathematical judgment goes beyond the apprehension of sense. So it is that sometimes a judgment we make about a mathematical line is not the same as one we make about a sensible line, for example, that a straight line touches a sphere at only one

[58] Maurer, "Thomists and Thomas Aquinas on the Foundation of Mathematics," 58–9. Emphasis mine. More recently, Adam Drozdek echoes the same sentiment: "If infinity is God's attribute and if it also can be found in the world, then there is a distinct difference between these two kinds of infinity, to the extent that the infinity to be found in the world is really not infinity at all (as in Wyclif). But infinity does not necessarily have to be God's attribute. If it is not, infinity can have free reign in the world; it can actually exist and yet not contradict the supremacy of God since he is above infinity—he is the Absolute. This is the stance taken by Cantor, and it is quite possible that Thomas himself, had he known Cantor's theoretical accomplishments, would not have opposed it." Drozdek, "Number and Infinity: Thomas and Cantor," 46.

> point, which is true of a separate straight line, but not of a straight line in matter, as is said in the first book of *De Anima*.[59]

Maurer and Anderson have a point. If, Aquinas and Aristotle see a marked difference in what we can assert of sensible, physical lines, as opposed to the mathematical lines we have abstracted—which they do, as we observe in the *precision-problem*—then the *ready* conclusion we drew above may have been premature. Indeed, the issue may only indirectly concern the imagination, for if mathematical lines can be shown to have properties which differ from their physical counterparts, then there is no need for mathematical objects to be legitimized beyond establishing that their inclusion leads to no *internal* inconsistencies. This is something both Aristotle[60] and Thomas Aquinas admit. Still, Anderson's conclusion is tentative, at least as far as Aristotle is concerned.

> Our question is then is the mathematician free to use these basic elements to construct (and hence demonstrate the existence of) any figure he desires—any figure that is, whose very existence is not self-contradictory (like square circles)? Certainly the most basic abstracted elements, those whose existence is simply posited, are so general as to be able to form any figure or number. And yet the Stagirite never states that the mathematician has the freedom to construct these elements into any non-self-contradictory objects he pleases.

[59] *In De Trinitate*, VI 2 corpus. "Quaedam vero sunt, quorum iudicium non dependet ex his quae sensu percipiuntur, quia quamvis secundum esse sint in materia sensibili, tamen secundum rationem diffinitivam sunt a materia sensibili abstracta. Iudicium autem de unaquaque re potissime fit secundum eius diffinitivam rationem. Sed quia secundum rationem diffinitivam non abstrahunt a qualibet materia, sed solum a sensibili et remotis sensibilibus condicionibus remanet aliquid imaginabile, ideo in talibus oportet quod iudicium sumatur secundum id quod imaginatio demonstrat. Huiusmodi autem sunt mathematica. Et ideo in mathematicis oportet cognitionem secundum iudicium terminari ad imaginationem, non ad sensum, quia iudicium mathematicum superat apprehensionem sensus. Unde non est idem iudicium quandoque de linea mathematica quod est de linea sensibili, sicut in hoc quod recta linea tangit sphaeram solum secundum punctum, quod convenit rectae lineae separatae, non autem rectae lineae in materia, ut dicitur in I de anima." This translation is mine, as are all subsequent translations from Thomas. Unlike Anderson, Aquinas does not appear to associate the imagination with mathematical freedom. Importantly, both Aristotle and Thomas differentiate the representational function of the imagination from its creative function. Anderson seems to be assuming that it is the creative imagination which is at issue when speaking of mathematics. Indeed, the creative imagination can be a source of falsehood in mathematics. See George, "Imagination a Source of Falsehood According to Aquinas," especially 193–4, 196.

[60] Aristotle, *De Anima*, (Oxford: Oxford University, 1979), I 1 403a10–15.

> In fact, it is just the opposite as we have said, the only objects of geometry he cites are those which closely resemble physical magnitudes. Could this indicate that he never thought of allowing the mathematician freedom to construct and treat objects not resembling quantified physical things? On the other hand, it might be suggested that Aristotle would never have intended such a limitation of mathematics since numbers by their very nature as more abstract than magnitudes are clearly not able to be closely bound to physical quantities.[61]

Beyond the place of the creative imagination in Thomas Aquinas is the *disputed question* from Aquinas' *Commentary on the Sentences of Peter Lombard*, introduced above.[62]

> Conceptions in the intellect have a threefold relation to something outside the soul. For sometimes the intellect conceives the likeness of a thing existing outside the soul (for example, what it conceives about the word 'man'), and such a conception of the intellect has a foundation in the thing immediately, insofar as that very thing, through its conformity with the intellect, causes the intellect to be true and the name signifying that thing to be properly said of it. At other times, what the name signifies is not a likeness of a thing existing outside the soul, rather, it is something which follows upon the way [the intellect] understands a thing outside the soul: and the intentions which our intellect discovers are of this sort (for example, what the name 'genus' signifies is not a likeness of some thing existing outside the soul, rather, because the intellect understands 'animal' as divided into many species, it attributes the intention of 'genus' to it). Now, although the proximate foundation of this sort of intention is not in the thing but in the intellect, still, its remote foundation is the thing itself. And so the intellect which discovers these intentions is not false. And so it is with all other things which follow upon the mode of understanding, such as the abstraction of mathematicals, and the like. At still other times, what a word signifies has neither a proximate nor a remote foundation in a thing (for example the concept of a Chimera, since it is neither a likeness of some thing existing outside the

[61] Anderson, "Aristotle and Aquinas on the Freedom of the Mathematician," 238.

[62] Maurer, "A Neglected Thomistic Text on the Foundation of Mathematics," 185–92. See also Maurer, "Thomists and Thomas Aquinas on the Foundation of Mathematics," 43–61. Antoine Dondaine established the origin and authenticity of this *quaestio* in "Saint Thomas et La Dispute des Attributs Divins (I Sent. d. 2, a. 3): Authenticité et Origine,"

> soul, nor does it follow upon [the intellect's] way of understanding a natural thing): and therefore this conception is false.[63]

If Aquinas consistently sees the mathematical sciences as concerned only with abstractions having direct correlates to physical aspects of realities existing outside the mind, then why would he have grouped mathematical abstractions along with the *logical* intentions (to use Aquinas' example, the notion of genus)? If a geometer considers the physical circle, though abstractly, as was said, then why would he not have classified such conceptions along with those having an immediate foundation in 'a thing existing outside the soul'? The conclusion follows quite readily.

> From one point of view, therefore, the objects of mathematics are real, but from another, and perhaps more significant point of view they are devised by the intellect. … mathematical abstractions are not totally independent of experience, nor are they found ready made, so to speak, in reality. The importance of the *Scriptum* text lies in the fact that it shows St. Thomas' awareness of the central role in mathematics of the *creative* work of the intellect, by locating the immediate foundation of mathematics not in reality but in the activity of the intellect.[64]

[63] *Scriptum Super Sententiarum*, I 2 3. "…ipsa conceptio intellectus tripliciter se habet ad rem quae est extra animam. Aliquando enim hoc quod intellectus concipit, est similitudo rei existentis extra animam, sicut hoc quod concipitur de hoc nomine homo; et talis conceptio intellectus habet fundamentum in re immediate, inquantum res ipsa, ex sua conformitate ad intellectum, facit quod intellectus sit verus, et quod nomen significans illum intellectum, proprie de re dicatur. Aliquando autem hoc quod significat nomen non est similitudo rei existentis extra animam, sed est aliquid quod consequitur ex modo intelligendi rem quae est extra animam: et hujusmodi sunt intentiones quas intellectus noster adinvenit; sicut significatum hujus nominis genus non est similitudo alicujus rei extra animam existentis; sed ex hoc quod intellectus intelligit animal ut in pluribus speciebus, attribuit ei intentionem generis; et hujusmodi intentionis licet proximum fundamentum non sit in re sed in intellectu, tamen remotum fundamentum est res ipsa. Unde intellectus non est falsus, qui has intentiones adinvenit. Et simile est de omnibus aliis qui consequuntur ex modo intelligendi, sicut est abstractio mathematicorum et hujusmodi. Aliquando vero id quod significatur per nomen, non habet fundamentum in re, neque proximum neque remotum, sicut conceptio Chimerae: quia neque est similitudo alicujus rei extra animam, neque consequitur ex modo intelligendi rem aliquam naturae: et ideo ista conceptio est falsa."

[64] Maurer, "A Neglected Thomistic Text on the Foundation of Mathematics," 192. Emphasis mine.

Maurer develops this account further in his article "Thomists and Thomas Aquinas on the Foundation of Mathematics," a sort of *status quaestionis* for those interested in Thomas Aquinas' philosophy of mathematics. He begins by summarizing accounts given by Joseph Gredt, Jacques Maritain, Edward Maziarz, Yves Simon, Thomas Anderson, and Vincent Smith, as to Aquinas' view of the nature and reality of mathematical objects. Though Thomists—in the broad sense—they do not agree.[65] Some incline more toward saying that recent developments in mathematics are consistent fictions, while others are comfortable with saying that Aquinas, too, would have judged objects resulting from the creative imagination to be true and legitimate. In the end, Maurer himself, like Anderson, is more liberal in his view.

> There are natural lines and circles but these are not the lines and circles of mathematics. There are also numbered or numerable multitudes in reality but they are not the numbers of mathematics. The objects of mathematics, though remotely based on the sensible world and abstracted from it, are products of a mental activity which alters their definitions and natures. Nothing like them exists in the external world. Aquinas was clearly aware of this from his knowledge of contemporary mathematics. Had he known modern types of mathematics, he would have seen *the almost unlimited range of the mind's mathematical inventiveness*.[66]

If Aquinas goes beyond Aristotle in grounding mathematical judgments, not in the senses but in the imagination, then why not allow legitimacy to whatever mathematical objects we can imagine? If, as both Anderson and Maurer insist, he ascribes to mathematical concepts a status not unlike those of the logical intentions, *entia rationis*, why place any restrictions upon mathematical discoveries other than that such constructs be self-consistent?

Recall Benacerraf's challenge to mathematical realism: one could either give an account of the extra-mental existence of numbers or supply a consistent epistemology for human knowledge, not both. A realist account would inevitably founder in its attempt to explain how mathematical statements are true and false in the same way non-mathematical statements are,

[65] "From this brief survey of some of the leading Thomistic interpretations of Aquinas's conception of mathematics it is readily apparent that there are significant differences among them." Maurer, "Thomists and Thomas Aquinas on the Foundation of Mathematics," 47.

[66] Ibid., 58–59. Emphasis mine.

or it would succeed there, but be unable to account for the existence of non-empirical, abstract *entities*. Recall that Benacerraf's own proposal was a specific form of mathematical *nonrealism*.[67]

I believe the attempts Anderson and Maurer make at reconciling the clear sense of Aristotle and Aquinas philosophy of mathematics with modern and contemporary developments to mathematics do not provide an adequate account of mathematical truth—they run afoul of the *epistemological* horn of Benacerraf's dilemma. Given the freedom now afforded mathematical investigations, *how* might we establish that some mathematical constructions are false and others true?

Recall also that Aquinas' threefold distinction—a key part of Anderson and Maurer's argument—is in place precisely to distinguish true conceptions from false ones. If a remote connection to things existing outside the mind sufficiently distinguishes a true conception from a false, and all mathematical constructs *ipso facto* possess such a connection, then all would be *ipso facto* true. (Even Cantor, who argued vehemently for the legitimacy of his transfinite numbers, gave to *metaphysics* the task of discovering their transient relations to the external world.) What of otherwise consistent mathematical systems which contradict *one another*?[68] On the other hand, Anderson and Maurer might, like Cantor, further insist that subsequent developments not contradict mathematical concepts *already established*.[69] They do not say. Still, we can ask the question ourselves: are the restrictions traditional philosophy would seemingly place upon mathematical inquiry justified? Are Cantor's *scholastic fears* warranted? Some make the case that they are not, at least as far as Aristotle and Thomas

[67] Benacerraf, "What Numbers Could Not Be," 47–73 and "Mathematical Truth," 661–79.

[68] Anderson includes non-Euclidean geometries among the legitimate developments of mathematics. Presumably he would include Euclidean geometry as well. See Anderson, "Aristotle and Aquinas on the Freedom of the Mathematician," 238.

[69] As Cantor says: "The procedure for the correct construction of concepts is in my opinion everywhere the same; one sets up a thing [Ding] having no properties, which at first is nothing more than a name or sign A and gives this according to some law, different and even an infinite number of properties, whose meaning is known through ideas already existent which may not contradict one another; in this way the relations of A to concepts already existent is determined; if the process is complete, then all the conditions for awakening the concept A [zur Weckung des Begriffes A] which was slumbering within us, are present and it emerges into being [Dasein], equipped with intrasubjective reality, which is all that can be demanded of concepts; to establish its transient meaning is then the business of metaphysics. Cantor, *On Infinite, Linear Point-Manifolds*, 148–9, note 7 of §8. There is an irony to Cantor's words "if the process is complete" when seen in the light of Kurt Gödel's undecidability theorems.

Aquinas are concerned. Even if extensions to number theory beyond the natural numbers are not required to have physical correlates,[70] are they free of internal contradictions and consistent with previously established conceptions?

To this extent, then, mathematical operations freely, and successively, applied to the natural numbers provide an adequate explanation for constructing the numerical extensions. No doubt—and apart from what we have already seen in Aquinas' account of the threefold relation of concepts to things[71]—there are passages in both philosophers indicating that mathematical objects are not merely *discovered* in mathematical form in the external world, but are *constructed* by the mind. If this is so for mathematical objects in general, then why would they restrict the application of the arithmetical operations to the natural numbers? Is not our very notion of the number twenty-five tied to its being the square of five? Do we not actually arrive at the notion of the number two by adding the unit to itself—the simplest possible operation? Are not all numbers *made*, anyway? Why can we not make them as we please?

The disallowance of such extensions is based upon another principle altogether, for, as we have argued above, the answer lies in math's principle. "The unit is indivisible, whatever it may be (for example, a man is one man and not many,) but number is many ones and a certain quantity."[72] Aquinas, too, notes that "*number* signifies that there are many things in one."[73] It is of the essence of number that it be a plurality, yet not a mere plurality—it has an essential connection to unity as well. "*Plurality* is a sort of genus of number: for number is *a plurality measured by the unit*."[74]

[70] We do not take it as a given that they need not; we take it as obvious that they do not. We intend to make the case that even these minimal conditions for declaring such free constructions mathematically legitimate and admissible by the standards of traditional philosophy are not met.

[71] Note that Aquinas' distinction already implies a sort of creative act on the part of the intellect, as Anderson and Maurer make clear.

[72] *Physica*, III 7 207b5–7. "…τὸ ἕν ἐστιν ἀδιαίρετον, ὅ τι περ ἂν ἓν ᾖ (οἷον ἄνθρωπος εἷς ἄνθρωπος καὶ οὐ πολλοί), ὁ δ' ἀριθμός ἐστιν ἕνα πλείω καὶ πόσ' ἄττα…"

[73] *In Physicorum*, III 12 392. "Numerus enim hoc significat, quod sint aliqua plura uno."

[74] *Metaphysica*, X 7 1057a2–4. τὸ δὲ πλῆθος οἷον γένος ἐστὶ τοῦ ἀριθμοῦ: ἔστι γὰρ ἀριθμὸς πλῆθος ἐνὶ μετρητόν. See also *In Metaphysicorum*, X 8 2080. The emphases are mine. Aquinas refers to the unit as the principle of number in many places, especially when distinguishing the one which is convertible with being from the numerical unit, which adds to mere undivided being the *ratio* of being a measure.

If we seek a foundation for the *episteme* of numbers as traditional philosophy presents it, we need look no further than the numerical unit which, though not itself a number, is the measure of all numbers.[75] Note that this is *Cantor's* measure, too: if we are to legitimize developments to number theory on the basis of the only restrictions Cantor allows, we have to determine whether such constructions are self-consistent *and consistent with mathematical concepts already in place*. But the first such concept is the unit. The same would follow for creative extensions to the *episteme* of continuous quantity in three dimensions.

While Aristotle distinguishes art from science, and even distinguishes between the constructive and demonstrative aspects of a scientific proof, it is Aquinas who supplies an account of the *mathematical arts* as distinguished from the *mathematical sciences*.[76] No doubt, we *make* two by adding the unit to itself—even if *addition* is not of the nature of two. A number is not *essentially* a result of mathematical operations.

Beyond that, Aristotle and Thomas do not see artificial things as essential realities, meaning that they have no natures in and of themselves, but are merely identical to what composes them.[77] Legitimacy—again, for them—goes well beyond internal and external consistency.

The mathematical arts provide us with constructions (numbers and such) whose properties we investigate. Sometimes these constructions are legitimate, we have said, 'natural,' while at other times they are not. How can we decide which is which?

The legitimacy of any construct depends upon the nature of its principle. If we are scientifically to know the properties of numbers, we must first construct those necessitated by the arithmetical unit—*only these can be called numbers in the proper sense*. It is not sufficient merely to employ the tools of the art of arithmetic, one must also demonstrate that such constructs are self-consistent and consistent with concepts already-established, *a fortiori* with the ultimate subject and principle of the science itself. In this way alone can one assure scientific demonstration as opposed to

[75] Both Aristotle and Aquinas deny that the unit is a number. Note, too, that a principle can, but need not, be of a kind with that of which it is the principle. Matter, for example, is a principle both of substance and of change, but is itself neither substance nor change. On the other hand, the motion of the moon about the earth is a principle of the motion of the tides: yet each is an instance of a change in place.

[76] We summarize in brief here what we developed extensively in Chaps. 6 and 8.

[77] One clear *locus* for this distinction is *Physica*, II 1.

artistic fabrication.[78] Despite Anderson, Maurer, and Drozdek to the contrary, we think Aristotle and Thomas Aquinas would not be open to extensions in the system of numbers that are incompatible with its ultimate subject and first principle, the unit. The same holds for extensions in the study of continuous quantity in three dimensions. We have made the case above just how far such extensions can be legitimated.

So far, then, as we have ranged over the history of prominent mathematical views, we have discovered no contemporary view that maps to the view we have presented here: that mathematics consists of the *episteme* of arithmetic and the corresponding *episteme* of geometry. From Plato to *the big three* through anti-realisms and realisms—whether Aristotelian or Thomistic—there seems to be no account of mathematics which meets all the conditions we have found in the writings of Aristotle and Aquinas. None comports so closely as authentically Aristotelian and Thomistic as does the current account. What remains is to compare the narrow conception of *mathematics*—the *prime analogate*, one might say—with, and differentiate it from, its extended senses, which is our concern in the final chapter.

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[78] Once again the analogy to the five regular solids is apropos. It may be possible to construct other regular solids given the assumptions of non-Euclidean geometries: yet these are not the assumption of the three-dimensional continuum. This assumption entails these consequences.

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PART IV

Conclusion

CHAPTER 14

Mathematics Is a Mixed Bag

Nearly all the philosophers of mathematics we have read, considered, and quoted in this book have one question in common: what *is* mathematics? Like Augustine's inability to define with precision what is so obvious otherwise,[1] those who esteem mathematics—among which we must count these philosophers and many, many more (and who but a lover would remain with such a question through such twists and turns as these?) have read, considered, and quoted others in their own turn. That some people have been able to sift through sundry *accidental* connections to arrive at the essence of the matter is a great gift to us all. We believe that the mathematical observations of Aristotle and certain developments of those by Thomas Aquinas not only *drill down* to (or close to) the essence of what everyone knew mathematics to be, but also avoid the many difficulties encountered by other thinkers along the way.

As nearly as we can see, mathematics is *a mixed bag*. This may be a bit *generous* if we are speaking of whatever things have gone by the name of *mathematics* up to the present day. It's not at all clear that there is *some* commonality even among the *Big Three*—that is, it may be that they

[1] At issue for Augustine is the nature of *time*, "What, then, is time? I know well enough what it is, provided that nobody asks me; but if I am asked what it is and try to explain, I am baffled." Augustine, *Confessions*, trans. R. S. Pine-Coffin (New York: Penguin Books, 1961), XI 14.

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J. W. Rioux, *Thomas Aquinas' Mathematical Realism*,
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cannot even be placed into the same *bag*. On the other hand, if we limit *mathematics* to certain real and well-defined activities of the mind regarding their common subject, *mixed bag* seems an *underdetermination*, in that the term *mathematics* seems really to denote mental capacities surrounding the apprehension of quantified substances. When used properly and not metaphorically, *mathematics* signifies πρὸς ἕν.[2] Aristotle famously introduces this notion in the *Metaphysics*, when speaking of the subject of this form of *episteme*, *being as such*.

> *Being* is said in many ways—not univocally, but in relation to one [sense] and one certain nature. *The healthy* is related to *health*, one [sense] being *that which safeguards health*, another *that which causes it*, a third *that which signifies it*, and still another *that which can receive it*. So, too, *the medical* in relation to [the art of] *medicine* (for in one [sense] *the medical* is what *possesses [the art]*, in another what is *naturally related to it*, in a third it is *its very activity*. Other words are used in the same way. So, too, there are many ways in which *being* is said, yet all [the other senses] are said in relation to a single principle.[3]

All—even Russell[4]—are agreed that *mathematics* is a kind of knowing about something. But there are different ways to know,[5] first among which is *episteme*.[6]

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[2] *Pros hen* equivocation is a specific type of ambiguity, one rooted in a real difference among the things so named.

[3] *Metaphysica*, IV 2 1003a32–b23. τὸ δὲ ὂν λέγεται μὲν πολλαχῶς, ἀλλὰ πρὸς ἓν καὶ μίαν τινὰ φύσιν καὶ οὐχ ὁμωνύμως ἀλλ᾽ ὥσπερ καὶ τὸ ὑγιεινὸν ἅπαν πρὸς ὑγίειαν, τὸ μὲν τῷ φυλάττειν τὸ δὲ τῷ ποιεῖν τὸ δὲ τῷ σημεῖον εἶναι τῆς ὑγιείας τὸ δ᾽ ὅτι δεκτικὸν αὐτῆς, καὶ τὸ ἰατρικὸν πρὸς ἰατρικήν (τὸ μὲν γὰρ τῷ ἔχειν ἰατρικὴν λέγεται ἰατρικὸν τὸ δὲ τῷ εὐφυὲς εἶναι πρὸς αὐτὴν τὸ δὲ τῷ ἔργον εἶναι τῆς ἰατρικῆς), ὁμοιοτρόπως δὲ καὶ ἄλλα ληψόμεθα λεγόμενα τούτοις, οὕτω δὲ καὶ τὸ ὂν λέγεται πολλαχῶς μὲν ἀλλ᾽ ἅπαν πρὸς μίαν ἀρχήν.

[4] Bertrand Russell famously described mathematics as … *the subject in which we never know what we are talking about, nor whether what we are saying is true*. "Mysticism and Logic" in *Mysticism and Logic and Other Essays*, (London: Allen & Unwin, 1910), 75.

[5] Here again, Aristotle supplies a most helpful insight. That knowing is said in many ways is one of his most important contributions to material logic. We saw above how it was crucial to solving Meno's paradox. Here, it serves to explain why *mathematics* would be so many different things to different people.

[6] First, that is, not in the order of knowing (we certainly do not begin with *episteme* in our intellectual journeys) but in the order of perfection. Recall that it supplies that *deepest* form of explanation, as an answer to the question *why*. Among the forms of *episteme* is *sophia*, an understanding of the ultimate *reasons why*.

> …in all inquiries which have principles, causes, or elements, knowledge and *episteme* come about from apprehending these (for we think we know each of them when we know its first causes and first principles, all the way down to its very elements).[7]

While Aristotle uses this observation to introduce the general *episteme* of natural things, the same would apply to any of the various forms of *episteme*. Its very conditions—that one know the causes of a thing, that they are the causes, and that they cannot be otherwise[8]—require such a connection to the question *why*.

> Having determined these matters we ought to investigate the causes—both what and how many they are. For our inquiry is for the sake of knowledge, and we do not think that we know a thing until we grasp the *why* of it (which is to grasp its first cause).[9]

Apart from the question whether we are even *capable* of such depths in our knowledge of the world, it is reasonable to at least *expect* philosophers[10] to grant this point—such ultimate explanations are at least consistent with philosophy as it is commonly regarded. If this is so, then mathematical knowledge would also find its apex in *episteme*. Even our philosophical *temperaments* confirm this—evidence the relatively lesser regard we afford indirect proofs over direct ones. It is one thing to become convinced of the *fact of* the equality of a triangle's angles to one-hundred-eighty degrees, but that does not compare to the satisfaction of seeing it demonstrated in such an elegant way.[11] The *itch* of wonder is left unsatisfied with mathematical *facts* but requires mathematical *explanations*.

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[7] *Physica*, I 1 184a9–14. … τὸ εἰδέναι καὶ τὸ ἐπίστασθαι συμβαίνει περὶ πάσας τὰς μεθόδους, ὧν εἰσὶν ἀρχαὶ ἢ αἴτια ἢ στοιχεῖα, ἐκ τοῦ ταῦτα γνωρίζειν (τότε γὰρ οἰόμεθα γιγνώσκειν ἕκαστον, ὅταν τὰ αἴτια γνωρίσωμεν τὰ πρῶτα καὶ τὰς ἀρχὰς τὰς πρώτας καὶ μέχρι τῶν στοιχείων).

[8] Cf. *Analytica Posteriora*, I 2 71b8–12.

[9] *Physica*, II 3 194b17–20. Διωρισμένων δὲ τούτων ἐπισκεπτέον περὶ τῶν αἰτίων, ποῖά τε καὶ πόσα τὸν ἀριθμόν ἐστιν. ἐπεὶ γὰρ τοῦ εἰδέναι χάριν ἡ πραγματεία, εἰδέναι δὲ οὐ πρότερον οἰόμεθα ἕκαστον πρὶν ἂν λάβωμεν τὸ διὰ τί περὶ ἕκαστον (τοῦτο δ' ἐστὶ τὸ λαβεῖν τὴν πρώτην αἰτίαν).

[10] That is, lovers of wisdom.

[11] Cf. Euclid, *The Elements*, I, 32.

> For, as we said, we begin by wondering that things are as they are before we know their causes, as we do when wondering about *automatons* or the turnings of the sun (solstices) or the incommensurability of the diagonal—for it seems wonderful to all who do not know the reason that there is something which cannot be measured, even by the smallest part. But we must finish up in the opposite and—so says the proverb—better condition, as happens with those who learn [μάθωσιν]. For nothing would cause the geometer to wonder as much as if the diagonal turned out to be commensurable.[12]

Wonder drives mathematical inquiries, as it does all our endeavors to know. If the Greek thinkers passed along any wisdom to us—which, in my view, they surely did, and abundantly—it is that we can begin to learn upon recognizing that we do not know. The *episteme* of mathematics represents the ultimate attempt at slaking our intellectual thirst, to apprehend the riches of the seemingly mundane realm of the *how much*.

Extending our analogy of the mixed bag, then, we would regard the *episteme* of mathematics as one necessary inclusion, to be sure, but more. Being mathematical *knowing* in its most perfect sense, it is the one sense of the word *mathematical* in relation to which other senses get their meanings.

But what other senses are there? We discern at least four: as *practical* knowledge it is (1) a *method* for the mixed sciences as well as (2) a liberal art. We see also a sense in which it is (3) a *creative*—even a *fine*—art. Lastly, it is (4) a *heuristic*—one among many possible methods whereby to achieve a given result.

The Quine-Putnam indispensability argument attempts to secure the *realism* of mathematics on a foundation taken to be *antecedently real*, namely, the empirical sciences. The *episteme* of nature developed by Aristotle *via* the *Physics* and his other works in natural philosophy are notably different from post-Enlightenment sciences in at least one obvious way: they do not employ a mathematical method.[13] On the other hand,

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[12] *Metaphysica*, I 2 983a14–20. ἄρχονται μὲν γάρ, ὥσπερ εἴπομεν, ἀπὸ τοῦ θαυμάζειν πάντες εἰ οὕτως ἔχει, καθάπερ περὶ τῶν θαυμάτων ταὐτόματα τοῖς μήπω τεθεωρηκόσι τὴν αἰτίαν ἢ περὶ τὰς τοῦ ἡλίου τροπὰς ἢ τὴν τῆς διαμέτρου ἀσυμμετρίαν(θαυμαστὸν γὰρ εἶναι δοκεῖ πᾶσι τοῖς μήπω τεθεωρηκόσι τὴν αἰτίαν εἴ τι τῷ ἐλαχίστῳ μὴ μετρεῖται): δεῖ δὲ εἰς τοὐναντίον καὶ τὸ ἄμεινον κατὰ τὴν παροιμίαν ἀποτελευτῆσαι, καθάπερ καὶ ἐν τούτοις ὅταν μάθωσιν: οὐθὲν γὰρ ἂν οὕτως θαυμάσειεν ἀνὴρ γεωμετρικὸς ὡς εἰ γένοιτο ἡ διάμετρος μετρητή.

[13] For the most part.

modern science *as practiced* is unthinkable[14] apart from it. This is where Aristotle's account of the *middle sciences* comes in. If there is an *episteme* of mathematics, then mathematicals are to be found among physical substances *in principle*. While one abstracts from the *physicality* of these substances in doing mathematics proper, it remains that the physical universe is mathematically arranged, *in concreto*. No wonder that theorems—beginning and ending with *abstract* considerations of quantity—find an application among those very things from which our mathematical apprehension arose. This is another sense of *mathematics* then: *mathematics as applied*. Its place *in the middle* is easy to see. It is not *pure*, so to speak, in that its very *raison d'etre* is not to put our philosophical wonderings about quantity to rest, but to bring about a good result, which could itself be either speculative or practical.

Take optics. Were we to wonder why it is that our mirror image is a *mirror* image, geometry—as instanced in the propagation of light—brings our questioning to rest. Here, one applies mathematical theorems to a physical reality to achieve *episteme*—itself a *pure* knowledge.[15] But the conclusions of optics can themselves be *applied* so as to bring about some definite *physical* result, such as constructing a mirror whose reflection is not a *mirror* image.[16] Another—albeit approximate—term for this fruitful interchange among the sciences is *subalternation*.[17]

That *mathematics* as *applied* differs from *pure mathematics* is a commonplace. To those who see the empirical sciences as presenting a true view of the world—scientific realists—it is important that mathematics be *first an episteme*. After all, to what extent is the *law of reflection* true if there really are no such things as angles? On the other hand, scientific nonrealists—so-called *instrumentalists*—have no concern as to whether the

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[14] *Pace* Hartry Field.

[15] For its part, natural philosophy does not find an application to mathematics, since it is would be applying those very apprehensions and truths *from which mathematics—to be mathematics—abstracts* in doing so, whereas mathematics brings to a consideration of sensible substances something which is *already present in them*.

[16] Assuming that is even possible. Even if not, its impossibility would be discovered by the application of theorems in optics which are the result of the theorems of pure geometry.

[17] The terms are closely related but not interchangeable. See Bernard Mullahy, "Subalternation and Mathematical Physics," *Laval Théologique et Philosophique* 2 no. 2 (1946): 89–107. Mullahy's article is an excellent source for the distinction between *episteme* and mathematics in application.

models of pure mathematics are descriptive *in re*—after all, even the *scientific models* achieved through their application are regarded as *mere* models.[18]

Different thinkers, then, take *applied mathematics* quite differently, depending upon what they intend in the application. If knowing *why* is the goal, it matters that the mathematics which is applied be an *explanation*. If not, not.

To be fair, there are such things as—to borrow Field's term—*useful fictions*. It is unlikely that many empirical scientists regard any aspect of the physical universe as being *i*-like, though *i* is *a crucial part* of certain physical descriptions. Similarly, a story is told about a modern-day sailor whose responsibilities included tidal predictions. When questioned about the *geostatic model* he used to do so, he is said to have responded: "Yes, I know. But it's just easier that way." It is far easier to render balance sheets on the assumption of negative numbers than not, just as it is far easier to speak of fractional parts when stress-testing materials and continuous changes when using satellite positioning. On the other hand, round-off errors *are* errors, and approximations—though *good enough*—are still not *good*.

As we think the notion of applied-mathematics-as-mere-model deserves its own consideration, we next turn to our *third* sense of the word, mathematics as *art*.

The notion central to any art is maintained in our current use of the associated term, *technology*.[19] While connoting *the more sophisticated and complicated* among them, the term really does apply to anything which is the product of human making. Aristotle describes the source of such products—art—as a capacity within us, *right reason as regards making things*. As we pointed out above, the traditional *liberal* arts are divided into two groups: the *trivium* and the *quadrivium*.[20] The arts constituting the *trivium* are (roughly) linguistic, while the four arts of the *quadrivium* are mathematical: *arithmetic*, *geometry*, *astronomy*, and *music*. As Aristotle is careful to distinguish *episteme* from *techne* when he gives an account of these *capacities within*, one might wonder how the same *knowing* could be classified as both. Which, of course, is precisely true. *Mathematics* as

[18] To illustrate, it is of great concern to the physical realist that spacetime *be* non-Euclidean in the specified manner, whereas the concern of the physical non-realist is only to discover that geometrical model of space—instanced or not—which produces correct predictions most consistently.

[19] Art is *techne*–τέχνη—in Greek.

[20] Literally, *the three ways* and *the four ways*, respectively.

episteme is not the same acquired capacity as *mathematics* the *art*. Their ends differ—one is a capacity to *demonstrate why*, the other is a capacity to produce a human artifact. We know all about the *episteme*, but what do the arts of mathematics *make*?

As we saw earlier, Aristotle and Thomas Aquinas have a solution to the *precision-problem*, to which, honestly, they are especially prone. That solution lies in distinguishing the ultimate subjects of mathematics—continuous quantity in three dimensions and the mathematical unit—from those properties we mentally construct and about which we go on to establish properties of their own. It is no accident that many of the *theorems* one finds in Euclid's *Elements* end with the initialism QEF—*quod erat faciendum*.[21] We pointed out earlier that even some of the *postulates* in geometry are *makings*. The art of construction goes hand-in-hand with the *episteme* of mathematics. After all, as part of the very requirements of *episteme*, one must first know that the *subject* of a conclusion *exists* before one can demonstrate anything about it. Construction-proofs are mathematical existence-proofs—demonstrations that a given mathematical composition necessarily result from, and is consistent with, mathematical principles. As we saw with Aquinas earlier, what sensation is to natural philosophy, the constructive imagination is to mathematics. It is here—among the liberal arts of mathematics—that the groundwork necessary for any perfect study of quantity must occur. The mathematical arts are also the gatekeeper, so to speak, to the mathematical *episteme*. Just as a demonstrated constructability yields a legitimate mathematical subject for which to find the reason why, a demonstrated inconstructability achieves the opposite: for there is no science of things that do not exist.

Imagination is often wrongly pointed to as the line-of-demarcation between fiction and fact. Imaginings are *fanciful* and *fantastic*. There are no *a priori* restrictions as to what we can imagine, whereas reality *is* a restriction when it comes to what is true. This is why the mathematical constructions of the liberal arts are not *mere imaginings*. Note that Aristotle includes *right reason* in his definition of *art—right reason as regards making things*. Mental constructions are presumed to happen by the action of our imaginations. That much seems obvious. That we would take our imaginings as being *true* or as being *fictions* does not stem from the imagination, properly and as such. The criterion—the *line-of-demarcation*—must originate elsewhere. It is obviously in conjunction

[21] *Which is what had to be made.*

with right reason, then, that a proper use of our imaginations lies. This applies to *any* artistic use of the imagination. But there is a difference.

Fictitious imaginings we *know* to be fictitious are *creative*. Right reason has its place in making them—but right reason must *first* know they are fictions. There is no *art* of confusing the world of fiction with reality. That is, rather, a psychotic *disorder*, delusion. The *rational* maker—and there *is* no other sort—must be first aware of the nature of the making, and the imagination is a powerful means to that end.

The use of the imagination with a view to *true* makings differs from its use with a view to *fictitious* makings. The one pertains to the liberal arts, the other to the creative, or even fine, arts.

Are fictitious mathematical makings legitimate? Apart from their utility—which we have addressed, above—it is hard to say, speaking precisely, that they are not. After all, how could one possibly say that a work of fiction is *illegitimate*, in some absolute sense? A making is what it is, no more, no less. What matters is that we recognize the *kind* of making it is. It is as tiresome to call into question the legitimacy of mathematical works of (pure) art on the grounds that they are fictions as to call into question the legitimacy of triangles on the grounds that they are not. The legitimacy of each is the result of *right reason* in the making. In discovering the incompatibility between *i* and real and true mathematical principles, right reason cannot then declare its compatibility and still be *right*. Similarly, in discovering that real and true mathematical principles suffice for the constructive demonstration of an equilateral triangle, right reason would be untrue to itself were it to call its reality into question.

Mathematical constructability through the imagination, then, can be *liberal* or merely *creative*—both fall within the scope of right reasoning.

Assuming that the products of the *fine* arts are (*precisely speaking*) fictions, the question arises whether we might even regard mathematical constructions as *works of art* in this common sense. The realm of the beautiful is vast, and extends far beyond what human imaginations, in conjunction with reason, can produce.[22] To the degree that mathematical constructs, bodies of proofs, and even individual theorems admit of beauty—and we take it as a matter of common experience that they

[22] Aristotle observes that, given the beauty of the painter's or sculptor's creations, one would naturally also find beauty in the "original realities" after which they are patterned. Aristotle, *De Partibus Animalium* (Oxford: Oxford University Press, 1986), I 1.

do—there does seem to be a sense in which mathematics can be taken as one of the *arts of the beautiful*.

Our final consideration has to do with a remaining question, which is, what of mathematical models which are assumed false, yet which produce consistent results—the counterpart to applying the *episteme* of mathematics to physical problems?

In the *Gorgias*, Socrates takes up a question regarding the art of *rhetoric* or *oratory*. At issue is whether it is a *true* art or merely a *knack—empeiria*, experience. By Socrates' account, the difference is that a knack aims at a short-term good, like pleasure, whereas a true art takes into account the nature of what it is perfecting. A use of rhetoric to persuade another to evil, then, would be a knack, aiming at the short-term goal of mere persuasion and not the truth. Persuasion, of course, is not (in itself) perfective, since one can also be persuaded of falsehood.

There is an analogous situation when we apply fictitious mathematical constructions and models. We call such a use *heuristic*, which is to say one is employing a method toward a determinate result, either prescinding from the question whether it is or is not a legitimate method, or even knowing that it is illegitimate but adequate to the short-term goal. One good example is *haggling*. There is no expectation (on either side) that the process of haggling determines an objectively fair price. Yet a fair price is exactly what (good-willed) participants want. The *actual* means of arriving at a fair price, however, is difficult and time consuming. What haggling achieves is the *estimation*, on the part of each party to the transaction, that 'I have been fairly dealt with.' Of course, a price known to be objectively fair price would (or should) produce that same estimation, though the converse is not true. Still, we haggle. It is a heuristic.

Certain applications in mathematics seem to be *heuristic*, as described. We mentioned i, above, as an instance. The tedium of constantly reducing i to its nonimaginary roots—especially if its use seems not to compromise the truth of the matter under consideration—is a high price to pay. It is likewise possible to render balance sheets without including negative numbers in the calculations, or to recast stress tests in terms of ever-changing units so as to avoid fractional numbers. Still, what a high price to pay, especially if we do not thereby compromise the truth. *Mathematical heuristics*, then, is another sense in which we commonly use the term.

In conclusion, again, mathematics is a mixed bag. Perhaps some of the ancient and recent confusion surrounding its foundations can be traced back to this reality. Just as the sultan's blind advisors returned from their

travels with the conviction that *an elephant is like a snake* or *an elephant is like a whip* or *an elephant is like a wall*, perhaps some philosophers of mathematics have sought to reduce the whole of it to a single reality, formula, or expression, to one *kind* of knowing or to knowing in only a *certain* way. Finally, then, to know *is* said in many ways, and this applies as much to knowing mathematically as to any other area.

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[1] Note: Page numbers followed by 'n' refer to notes.

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